

Repeated quantum non-demolition measurements: convergence and continuous-time limit

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Abstract

We analyze general enough models of repeated indirect measurements in which a quantum system interacts repeatedly with randomly chosen probes on which Von Neumann direct measurements are performed. We prove, under suitable hypotheses, that the system state probability distribution converges after a large number of repeated indirect measurements, in a way compatible with quantum wave function collapse. Similarly a modified version of the system density matrix converges. We show that the convergence is exponential with a rate given by some relevant mean relative entropies. We also prove that, under appropriate rescaling of the system and probe interactions, the state probability distribution and the system density matrix are solutions of stochastic differential equations modeling continuous-time quantum measurements. We analyze the large time convergence of these continuous-time processes and prove convergence.

1 Introduction

Repeated indirect quantum measurements aim at getting (partial) information on quantum systems with minimal impact on it. A possibility consists in repeating non-demolition measurements (QND). At each step, one lets the quantum system under study interact with another quantum system, called the probe, and performs a Von Neumann measurement[4] on this probe. Information on the quantum system is gained through intrication between the probe and the quantum system. If one is aiming at progressively measuring

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a quantum observable, one has to make sure that a system prepared in one eigenstate of this observable remains in it after a cycle of intrication and direct measurement on the probe, and that the set of stable states, called pointer states, forms an orthonormal basis of the system Hilbert space. The experiment of ref.[2], in which repeated QND measurements is used to fix and measure the number of photons in a cavity without destroying them, illustrates this strategy.

Repeated indirect measurements were studied in ref.[1]. There, the discussion was limited to QND measurements consisting of identical probes, interactions and measurements on the probe, and assuming a non-degeneracy condition. In the present article, we extend these results to cases where different indirect measurements (probes, interactions and direct measurements on probes) are used. We also study the degenerate case.

Let $Q_n(\cdot)$ be the pointer state distribution after the n^{th} indirect measurement, that is $Q_n(\alpha)$ is the probability to find the system in the state α after n steps (α labels the pointer states), $\sum_{\alpha} Q_n(\alpha) = 1$. As explain in section 3, each cycle of indirect measurement updates the distribution through Bayes' law. The analysis of the distribution is reformulated as a problem in classical probability theory (with no quantum interference).

We shall prove that this sequence of distributions converges at large n , that is after a large – strictly speaking, infinite – number of QND measurements. If a non-degeneracy assumption is verified, the limit distribution is $Q(\alpha) = \delta_{\Upsilon, \alpha}$ for some random limit pointer state Υ . This reflects the collapse of the system wave function. The convergence is exponential,

$$Q_n(\alpha) \simeq \text{const.} \exp(-n\overline{S}(\Upsilon|\alpha)), \quad \text{for large } n, \alpha \neq \Upsilon,$$

with rate given by an appropriate relative entropy $\overline{S}(\Upsilon|\alpha)$ defined in eq.(9). In probabilistic terms, the limit $Q(\alpha)$ possesses a natural interpretation as a Radon-Nikodym derivative and, $Q_n(\alpha) = \mathbb{E}(Q(\alpha)|\mathcal{F}_n)$ is a closed martingale with respect to an appropriate filtration, see section 5. As a consequence, we show that the expectations conditioned on the limit pointer state are identical to expectations starting from this same pointer state. That is: $\mathbb{E}_{\Upsilon}(\cdot) = \mathbb{E}(\cdot|\mathcal{A})$ where \mathcal{A} is the tail σ -algebra, the smallest one making the limit distribution measurable. See below for a precise definition of $\mathbb{E}_{\Upsilon}(\cdot)$.

Convergence of $Q_n(\alpha)$ is also studied when the non-degeneracy hypothesis is not fulfilled. In this degenerate case, the limit pointer state distribution vanishes outside a random finite set of pointer states. The quantum system density matrix, when properly modified, also converges in the limit of infinite number of QND measurements. The limit density matrix then coincides with that predicted for degenerate Von Neumann measurements[4], see section 5.

Of course repeated indirect measurements have already been studied in the physics literature, mostly through time continuous measurement formalisms – as far as we know, little was done on the discrete setting as we

do in the present paper. E. B. Davies[6] probably made the first rigorous approach to time continuous quantum measurement. This was later studied by N. Gisin[7] and L. Diosi[8] using the non linear Schrödinger equation. Simultaneously, A. Barchielli and V. P. Belavkin derived the equations governing continuous measurements in terms of instruments [9]. They derived jump equations which, when properly rescaled, are equivalent to diffusive equations for continuous measurements. Another approach uses quantum stochastic differential equations and quantum filtering theory to obtain the so-called Belavkin equations [10, 11, 12, 13, 14] and [15]. More recently, C. Pellegrini derived Belavkin equations for continuous time measurements [20, 21, 22] using discrete repeated indirect measurement models. The problem of convergence of quantum density matrix has also been analyzed within the time continuous measurement framework. In refs.[16, 17], V. P. Belavkin showed the convergence of mixed states toward pure states. A derivation of wave function collapse from the non-linear stochastic Schrödinger equation has been presented in refs.[18, 19]. It makes use of martingale theory as we do in the present paper.

In the following we also connect our discrete model to the time continuous measurement formalism. Taking the time continuous limit requires rescaling appropriately the interaction between the quantum system under study and the probes. In that sense, the time continuous model we consider is close to that of ref.[21] but our proofs are different and slightly more general. Our derivation is based on the convergence of some discrete counting processes – related to the number of occurrences of outputs in the successive indirect measurements – toward a time continuous Gaussian process. Under appropriate hypotheses, spelled out in section 6, the pointer state distribution satisfies a random diffusive stochastic equation driven by Gaussian processes. Suppose that at each step the probe system is randomly selected (independently of the past history and with time independent probability, for simplicity) among a finite set \mathcal{O} whose elements are indexed by $o \in \mathcal{O}$ and that the output measurements on the probe can take finite number of values indexed by $i \in \text{spec}(o)$. Then, the pointer state probabilities $Q_t(\alpha)$ are time continuous martingales (with respect to an appropriate filtration) whose evolutions are governed by the non linear stochastic equations:

$$dQ_t(\alpha) = Q_t(\alpha) \sum_{(o,i)} (\Gamma^{(o)}(i|\alpha) - \langle \Gamma^{(o)}(i) \rangle_t) dX_t(o, i)$$

where $X_t(o, i)$ are some centered Gaussian processes, $\Gamma^{(o)}(i|\alpha)$ are coding for the probability of output probe measurement i within the probe system o conditional on the quantum system be prepared in the state α and $\langle \Gamma^{(o)}(i) \rangle_t = \sum_{\alpha} \Gamma^{(o)}(i|\alpha) Q_t(\alpha)$. The pointer state distribution again converges as a finite-dimensionnal bounded vector martingale. Under non-degeneracy assumptions, the limit distribution is again $Q(\alpha) = \delta_{\Upsilon, \alpha}$ and

the convergence is still exponential with a rate given by the scaling limit of the mean relative entropy.

These results extend to the system density matrix. In the time continuous scaling limit, the system density matrix is a solution of a diffusive Belavkin equation (23), as expected. Although not a martingale, properly modified, it converges to the density matrix predicted by Von Neumann measurement theory.

The article is organized as follow : In section 2 we define the repeated QND measurement process we study. In section 3 we establish the link with a classical random process in which the pointer state distribution is repeatedly updated through Bayes' law. In section 4 we prove the convergence of the pointer state distribution under some assumptions and we determine the convergence rate in general cases. In section 5 we extend these results to the degenerate case. Finally in section 6 we study the time continuous scaling limit of our model. Some technical details appearing along the article are postponed to appendices.

2 QND measurements as stochastic processes

The aim of this section is to describe the relation between repeated non-demolition measurements, positive operator valued measurements (POVM's) and classical stochastic processes.

2.1 Repeated indirect quantum measurements

Let us consider a quantum system with initial density matrix ρ . Repeated non-demolition measurements aim at getting indirectly information on the system (without demolishing it as a projective quantum measurement à la Von Neumann or a direct connection to a macroscopic apparatus might do).

To gain information, we let the system interact with another quantum system called *the probe*, and then perform a Von Neumann measurement on the probe. Assume the probe is initially in the pure state $|\Psi\rangle\langle\Psi|$. Let U be the unitary operator, acting on the tensor product Hilbert space $\mathcal{H}_{\text{sys.}} \otimes \mathcal{H}_{\text{probe}}$, coding for the interaction between the system and the probe. After interaction, the system and the probe are entangled. Their joint state is

$$U(\rho_0 \otimes |\Psi\rangle\langle\Psi|)U^\dagger$$

A perfect non-degenerate projective measurement is then performed on the probe. That is, one is measuring an observable with a non-degenerate spectrum $i \in \mathcal{I}$. Let $\{|i\rangle\}$ be the corresponding eigenbasis of $\mathcal{H}_{\text{probe}}$. If the output of the probe measurement is i , the system state is projected into

$$\rho'(i) = \frac{1}{\pi(i)} \langle i|U|\Psi\rangle\rho\langle\Psi|U^\dagger|i\rangle$$

because the probe and the system have been entangled. This projection occurs with probability

$$\pi(i) := \text{Tr}[\langle i|U|\Psi\rangle\rho\langle\Psi|U^\dagger|i\rangle].$$

We do not have to worry about cases where $\pi(i) = 0$, because these cases, almost surely, never happen.

The process of "interaction plus probe measurement" is an example of a positive operator valued measurement (POVM). Let us define operators M_i , acting on the system Hilbert space, by

$$M_i := \langle i|U|\Psi\rangle.$$

They satisfy $\sum_i M_i^\dagger M_i = \mathbb{I}_{\text{sys.}}$ as a consequence of the unitarity of U . After measurement with output i , the system density matrix $\rho'(i)$ can be written as

$$\rho'(i) = \frac{1}{\pi(i)} M_i \rho M_i^\dagger,$$

with $\pi(i) = \text{Tr}[M_i \rho M_i^\dagger]$. This characterizes a POVM.

Let us now assume that we repeat the process of "interaction plus probe measurement" ad libitum. As we shall see below, even for purely practical reasons, it is useful to keep the freedom of changing some or all features of the process. For instance, the experimenter might tune (or let fluctuate randomly, or tune but leaving a certain amount of randomness or ...) the initial state $|\Psi\rangle$ of the probe at each step. Or he/she might tune (or let fluctuate randomly, or ...) the interaction operator U at each step, for instance by playing on the time lapse that the probe spends close enough to the system to interact significantly with it. He/She might even tune (or ...) the type of probe (in particular the dimension of its Hilbert space) at each step. Finally, he/she might tune (or ...) the non-degenerate probe measurement (equivalently the $\mathcal{H}_{\text{probe}}$ basis made of its eigenvectors $\{|i\rangle\}$).

We let $|\Psi_n\rangle$, U_n , \mathcal{I}_n denote the initial state, interaction operator and set of possible outcomes of the n^{th} step. Setting $\rho_0 := \rho$, $\rho_1 := \rho'$ and so on, we get a random recursion equation, namely that for $i \in \mathcal{I}_n$:

$$\rho_n = \frac{1}{\pi_n(i)} M_i^{(n)} \rho_{n-1} M_i^{(n)\dagger} \quad (1)$$

with probability $\pi_n(i) = \text{Tr}[M_i^{(n)} \rho_{n-1} M_i^{(n)\dagger}]$ where $M_i^{(n)} = \langle i|U_n|\Psi_n\rangle$ (note that the meaning of the expectation itself, i.e. the Hilbert space with respect to which it is taken, might depend on n , however we may arrange to choose the \mathcal{I}_n 's so that i determines $\langle i|$ completely).

It is worth noticing that, under such an evolution, a pure state remains a pure state, that is: $|\phi_n\rangle = M_i^{(n)}|\phi_{n-1}\rangle/\|M_i^{(n)}|\phi_{n-1}\rangle\|$ with probability $\langle\phi_{n-1}|M_i^{(n)\dagger}M_i^{(n)}|\phi_{n-1}\rangle$. This case is included in that of density matrices.

Let us now specialize this scheme in such a way that it preserves a preferred basis of the system Hilbert space. That is, we assume there exists a fixed basis $\{|\alpha\rangle\}$ of $\mathcal{H}_{\text{sys.}}$ such that all interactions can be decomposed as

$$U_n := \sum_{\alpha} |\alpha\rangle\langle\alpha| \otimes U_n(\alpha) \quad (2)$$

where the $U_n(\alpha)$'s are unitary operators on $\mathcal{H}_{\text{probe}_n}$. The states $|\alpha\rangle$ are called *pointer states*. The density matrices $|\alpha\rangle\langle\alpha|$ with $|\alpha\rangle$ a pointer state are fixed points of the recursion relation (1).

The pointer states have to be eigenstates of the system Hamiltonian H_s for these indirect measurements to be quantum non-demolition (QND) measurements, since there is a waiting time between two successive indirect measurements during which the quantum system evolves freely. That is, $H_s = \sum_{\alpha} E_{\alpha} |\alpha\rangle\langle\alpha|$ where E_{α} is the energy of the pointer state $|\alpha\rangle$ for the free system.

After each indirect measurement one gains information on the system state. Repeating the process (infinitely) many times amounts (as we shall explain) to perform a measurement of a system observable whose eigenstates are the pointer states. This observable commutes with the system free evolution. A system in one of the pointer states remains unchanged by the successive indirect measurements.

It has been shown in ref.[1] that a system subject to repeated QND measurements as described above converges toward one of the pointer states. This convergence was only proved in the case where the probes, interactions and observables on the probes are all the same. A non-degeneracy hypothesis was also used. One of the present article aim is to generalize the convergence statements without those assumptions.

A word on terminology: we are going to name *partial measurement* one iteration of "interaction plus probe measurement" and *complete measurement* an infinite sequence of successive partial measurements.

2.2 A toy model

We shall illustrate this framework and our results with a simple toy model inspired by experiments done on quantum electrodynamics in cavities[2]. The present work is actually inspired by these experiments. There, the system is a monochromatic photon field and the probes are modeled by two level systems. The observable we aim at measuring is the photon number. This is a non-demolishing measurement.

The system-probe interaction is well described by the unitary operator

$$U = \exp[-i(\epsilon\Delta t \hat{p} \otimes \mathbb{I}_2 + \frac{\pi}{4} \hat{p} \otimes \sigma_3)]$$

where \hat{p} is the photon number operator, ϵ the energy of a photon and Δt the interaction duration.

This interaction amounts to the rotation of the two level system effective spin half if the cavity happened to be in a photon number operator eigenstate. The probes are assumed to be initially in a state $|\Psi\rangle = e^{-i\theta\sigma_3}|+1\rangle$ where $|+1\rangle$ is an eigenvector of σ_1 corresponding to the eigenvalue $+1$. The probe observables, which are measured after the interaction between the system and the probe has taken place, are $O_{\theta'} = e^{-i\theta'\sigma_3}\sigma_2e^{i\theta'\sigma_3}$. Their eigenvectors are $|\pm_{\theta'}\rangle = e^{-i\theta'\sigma_3}|\pm_2\rangle$.

The resulting POVM operators for the process of "interaction plus probe measurement" only depend on the difference between the two angles θ and θ' . They are $M_{\pm}^{\theta-\theta'}$ with

$$\begin{aligned} M_{\pm}^{\theta-\theta'} &= \langle \pm_2 | e^{i\theta'\sigma_3} e^{-i(\epsilon\Delta t \hat{p} \otimes \mathbb{I}_2 + \frac{\pi}{4} \hat{p} \otimes \sigma_3)} e^{-i\theta\sigma_3} | +1 \rangle \\ &= e^{-i\epsilon\Delta t \hat{p}} \langle \pm_2 | e^{i(\theta'-\theta)\sigma_3 - i\frac{\pi}{4}\hat{p} \otimes \sigma_3} | +1 \rangle \end{aligned}$$

Using the identity $e^{-i\theta\sigma_3} = \cos(\theta)\mathbb{I}_2 - i\sin(\theta)\sigma_3$ and $\sigma_3|+1\rangle = |-1\rangle$, one gets

$$M_{\pm}^{\theta-\theta'} = \frac{1}{\sqrt{2}} e^{-i(\epsilon\Delta t \hat{p} \pm \frac{\pi}{4})} \left[\cos(\theta - \theta' + \frac{\pi}{4}\hat{p}) \pm \sin(\theta - \theta' + \frac{\pi}{4}\hat{p}) \right]$$

One may verify that $M_{+}^{\theta-\theta'} M_{+}^{\theta-\theta'}{}^\dagger + M_{-}^{\theta-\theta'} M_{-}^{\theta-\theta'}{}^\dagger = \mathbb{I}$. Remark that if $|p\rangle, p \in \mathbb{N}$ is a fixed photon number state, then $|\langle p | M_{\pm}^{\theta} | p \rangle|$ is identical to $|\langle p + 4k | M_{\pm}^{\theta} | p + 4k \rangle|$, with $k \in \mathbb{N}$. This property leads to degeneracies in iterative QND measurement methods. Two states $|p\rangle$ and $|p + 4k\rangle$ can not be distinguished by this method. These degeneracies are discussed in section 5.1.

3 Measurement apparatus and Bayes' law

The aim of this section is to reformulate (part of) of the iterative QND measurement method in terms of classical probability theory. We are interested in the eigenstate probability distribution $q_n(\alpha)$, with

$$q_n(\alpha) := \langle \alpha | \rho_n | \alpha \rangle,$$

and its evolution during the iterative procedure. At each step, the system density matrix is updated via the relation (1), and as a consequence of the factorization property (2), this distribution is updated through the random recursion relation:

$$q_n(\alpha) = q_{n-1}(\alpha) \frac{|M^{(n)}(i|\alpha)|^2}{\sum_{\beta} q_{n-1}(\beta) |M^{(n)}(i|\beta)|^2}$$

with probability $\sum_{\beta} q_{n-1}(\beta) |M^{(n)}(i|\beta)|^2$. We have defined $M^{(n)}(i|\alpha) := \langle i | U_n(\alpha) | \Psi_n \rangle$.

Since $|M^{(n)}(i|\alpha)|^2$ is the probability to get a probe measurement result i conditioned on the system state being $|\alpha\rangle$, we introduce a (hopefully) suggestive notation

$$p_n(i|\alpha) := |M^{(n)}(i|\alpha)|^2$$

We have $\sum_i p_n(i|\alpha) = 1$ since $\sum_i M^{(n)}(i|\alpha)^\dagger M^{(n)}(i|\alpha) = \mathbb{I}$. The recursion relation on the distribution reads

$$q_n(\alpha) = q_{n-1}(\alpha) \frac{p_n(i|\alpha)}{\sum_\beta q_{n-1}(\beta) p_n(i|\beta)} \quad (3)$$

with probability $\pi_n(i) = \sum_\beta q_{n-1}(\beta) p_n(i|\beta)$. This update rule corresponds to Bayes' law. The study of the eigenstate distribution convergence is thus a question of classical probability theory.

Let us now put on stage the classical probability theory framework we shall be using.

We imagine building a measurement apparatus which performs a sequence of partial measurements. As we have stressed above, we allow for a protocol where the characteristics of partial measurements may vary at each step. However, we shall assume that these characteristics are chosen within some finite set \mathcal{O} called the set of measurement methods. In a quantum setting, one measurement method o is a triplet (probe state $|\Psi\rangle$, interaction U , probe eigen-basis $\{|i\rangle\}$ of direct measurement). Each measurement method $o \in \mathcal{O}$ defines a set, called the spectrum of o and denoted by $\text{spec}(o)$, of possible outcomes. For each o we have a family of probability measures $p^o(\cdot|\alpha)$ on $\text{spec}(o)$ indexed by $\alpha \in \mathcal{S}$, where \mathcal{S} is the index set of pointer states.

As time goes by, the experimenter records the sequence $o_1, i_1, o_2, i_2, \dots$ where o_1 is the first measurement method, i_1 the outcome of the first partial measurement, performed using method o_1 , and so on. So it is natural to take as the space of events the space Ω of infinite sequences $(o_1, i_1, o_2, i_2, \dots)$ where each o_n belongs to \mathcal{O} and each i_n to $\text{spec}(o_n)$. To be even more formal, set $E := \cup_{o \in \mathcal{O}} \{o\} \times \text{spec}(o)$, so that E is the set of couples (o, i) with $o \in \mathcal{O}$ and $i \in \text{spec}(o)$. Then $\Omega := E^{\mathbb{N}^*}$.

For a finite sequence $(o_1, i_1, o_2, \dots, i_n, o_{n+1}) \in E^n \times \mathcal{O}$, $B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$ is defined as the subset of Ω made of all those ω 's whose first $2n+1$ components are $o_1, i_1, o_2, \dots, i_n, o_{n+1}$. We define $B_{o_1, i_1, o_2, \dots, o_n, i_n}$ analogously. We let \mathcal{F}_n be the σ -algebra generated by all the $B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$. Note that \mathcal{F}_0 is the σ -algebra generated by all the B_{o_1} , i.e. \mathcal{F}_0 codes for the first measurement method choice. For convenience we define $\mathcal{F}_{-1} \equiv \{\emptyset, \Omega\}$. Then $\mathbb{F} := (\mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1, \dots)$ is an increasing sequence of σ -algebras. We take \mathcal{F} to be the smallest σ -algebra on Ω containing all the \mathcal{F}_n , making $(\Omega, \mathcal{F}, \mathbb{F})$ a filtered measurable space. We could define another filtration by taking \mathcal{F}'_n to be the σ -algebra generated by all the $B_{o_1, i_1, o_2, \dots, o_n, i_n}$. While this may seem superficially a more natural choice of filtration, we

shall see below that \mathbb{F} is slightly more convenient. There is a natural collection of measurable functions on (Ω, \mathcal{F}) , namely the projections : for $\omega = (o_1, i_1, o_2, \dots, o_n, i_n, \dots)$ we set $O_n(\omega) = o_n$, $I_n(\omega) = i_n$. These can be used to define counting functions that play an important role in the following. We set $\epsilon_n(o, i) := \mathbb{1}_{O_n=o, I_n=i}$ and $N_n(o, i) := \sum_{1 \leq m \leq n} \epsilon_m(o, i)$ (with the usual empty sum convention $N_0(o, i) := 0$).

The first task is to put a probability measure on Ω . The next one will be to define a sequence of random variables on Ω solving the recursion relation (3).

If the measurement methods are given, the distributions of partial measurements are described by the $p^o(\cdot|\alpha)$. So what remains to be discussed is how the measurement methods are chosen, and we put the condition that this does not involve precognition. We suppose that a collection of non-negative numbers $d_{-1} = 1$, $d_0(o_1)$ (for all $o_1 \in \mathcal{O}$), \dots , $d_n(o_1, i_1, o_2, \dots, i_n, o_{n+1})$ (for all $o_1 \in \mathcal{O}$, $i_1 \in \text{spec}(o_1)$, \dots , $o_{n+1} \in \mathcal{O}$) is given in such a way that

$$\sum_{o_{n+1} \in \mathcal{O}} d_n(o_1, i_1, o_2, \dots, i_n, o_{n+1}) = d_{n-1}(o_1, i_1, o_2, \dots, i_{n-1}, o_n). \quad (4)$$

It is not difficult to produce such a collection. For instance if $c_0(\cdot)$ is a probability measure on \mathcal{O} and for each $n \geq 1$ $c_n(\cdot|o, i)$ is a probability measure on \mathcal{O} indexed by $(o, i) \in E$ then

$$d_n(o_1, i_1, o_2, \dots, i_n, o_{n+1}) \equiv c_0(o_1) c_1(o_2|o_1, i_1) \cdots c_n(o_{n+1}|o_n, i_n)$$

does the job. We call this special choice the Markovian feedback protocol. Some special cases are of interest. If we assume that for $n \geq 1$ $c_n(\cdot|o, i) = c_n(\cdot)$ does not depend on o, i , we arrive at something we call the random protocol. On the other hand, if b_0 is an element of \mathcal{O} and b_n , $n \geq 1$ a family of maps from E to \mathcal{O} , then taking $c_0(\cdot) := \delta_{\cdot, b_0}$ and, for $n \geq 1$, $c_n(\cdot|o, i) := \delta_{\cdot, b_n(o, i)}$ we arrive at the description of an experimenter deciding of the next measurement method by taking into account the previous measure outcome.

Given a collection of such non negative numbers $d_n(o_1, i_1, \dots, i_n, o_{n+1})$, and using the Kolmogorov extension theorem, it is easy to see that there is a unique probability measure \mathbb{P}_α on (Ω, \mathcal{F}) such that

$$\mathbb{P}_\alpha(B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}) = p^{o_1}(i_1|\alpha) \cdots p^{o_n}(i_n|\alpha) d_n(o_1, i_1, o_2, \dots, i_n, o_{n+1}).$$

Indeed, we see that the mandatory consistency condition is fulfilled : if the left-hand side is summed over o_{n+1} (using (4)) and then over i_n the formula for $\mathbb{P}_\alpha(B_{o_1, i_1, o_2, \dots, i_{n-1}, o_n})$ is recovered, which is needed since $B_{o_1, i_1, o_2, \dots, i_{n-1}, o_n}$ is the disjoint union of the $B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$ over the possible o_{n+1} and i_n . The normalization condition $\mathbb{P}_\alpha(\Omega) = 1$ and the positivity condition are obvious.

Note that in general, conditional on the sequence of measurement methods o_1, \dots, o_n , one has $\mathbb{P}_\alpha(i_1, \dots, i_n | o_1, \dots, o_n) \neq p^{o_1}(i_1 | \alpha) \dots p^{o_n}(i_n | \alpha)$. This is due to the feedback. For the cases when the d_n 's do not depend on the outcomes, in particular for the independent random protocol, equality is recovered.

We define also,

$$\mathbb{P} \equiv \sum_{\alpha \in \mathcal{S}} q_0(\alpha) \mathbb{P}_\alpha. \quad (5)$$

We use \mathbb{E}_α and \mathbb{E} to denote expectations with respect to \mathbb{P}_α and \mathbb{P} respectively.

A simple computation shows that, for each α , the conditional probability

$$\mathbb{P}_\alpha(O_{n+1} = o_{n+1} | O_1, I_1, \dots, O_n, I_n) = \frac{d_n(O_1, I_1, O_2, \dots, I_n, o_{n+1})}{d_{n-1}(O_1, I_1, 0_2, \dots, I_{n-1}, O_n)}$$

whenever the denominator is nonzero. The same formula holds for \mathbb{P} . The right-hand side is simply $c_n(o_{n+1} | O_n, I_n)$ for the Markovian feedback protocol. So indeed, the functions d_{-1}, d_0, d_1, \dots embody the probabilistic description of the choice of measurement methods.

These definitions may seem arbitrary at that point, but now we can make contact with the initial problem. Define a sequence of random variables $Q_n(\alpha)$ by the initial condition $Q_0(\alpha) = q_0(\alpha)$ and the recursion relation

$$Q_n(\alpha) = \frac{Q_{n-1}(\alpha) p^{O_n}(I_n | \alpha)}{\sum_{\beta \in \mathcal{S}} Q_{n-1}(\beta) p^{O_n}(I_n | \beta)}.$$

To show that the recursion relation (3) is verified, we need to show that the transition probabilities are correct. A simple way to do that is to solve this random recursion relation. For $\omega \in B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$ one checks that

$$Q_n(\alpha, \omega) := \frac{q_0(\alpha) p^{o_1}(i_1 | \alpha) \dots p^{o_n}(i_n | \alpha)}{\sum_{\beta \in \mathcal{S}} q_0(\beta) p^{o_1}(i_1 | \beta) \dots p^{o_n}(i_n | \beta)} \quad (6)$$

whenever $\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}) \neq 0$. Note that this condition ensures that the denominator of $Q_n(\alpha, \omega)$ is nonzero. We observe that whenever defined, $Q_n(\alpha, \omega) \geq 0$ and $\sum_{\alpha} Q_n(\alpha, \omega) = 1$. If $\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}) = 0$ the value of $Q_n(\alpha, \omega)$ is mostly immaterial from a probabilistic viewpoint, because in any case the full sequence $Q_n(\alpha, \omega)$ is well-defined on a set $\tilde{\Omega}$ of \mathbb{P} -measure 1 (note that the collection of all $B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$ is countable, so the collection of those with \mathbb{P} -measure 0 is countable as well, or empty).

Since there is no dependence on o_{n+1} on the right-hand side we observe that for $\omega \in \tilde{\Omega}$, and conditional on \mathcal{F}_{n-1} ,

$$Q_n(\alpha) = \frac{Q_{n-1}(\alpha) p^{o_n}(i_n | \alpha)}{\sum_{\beta \in \mathcal{S}} Q_{n-1}(\beta) p^{o_n}(i_n | \beta)}$$

with probability

$$\sum_{o_{n+1} \in \mathcal{O}} \frac{\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}})}{\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_{n-1}, o_n})} = \frac{\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_n})}{\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_{n-1}, o_n})} = \sum_{\beta \in \mathcal{S}} Q_{n-1}(\beta) p^{o_n}(i_n | \beta).$$

So the recursion relation (3) is recovered with the identifications $q_n(\alpha) \rightarrow Q_n(\alpha)$ and $p_n \rightarrow p^{o_n}$. To summarize, we have accomplished our goal : find a probability space on which (3) has a solution, which we have even written explicitly.

In the following sections we study the convergence of $Q_n(\cdot)$ and its dependence with respect to the initial pointer state distribution. On our way, we shall understand the probabilistic meaning of the recursion relation (3).

4 Convergence

In [1] the convergence of $q_n(\cdot)$ has been shown under the hypothesis that only one partial measurement method o is used. The properties of the limit were elucidated under the further assumption that for every couple of pointer states (α, β) there exists at least one partial measurement result i such that $p^o(i|\alpha) \neq p^o(i|\beta)$. This last assumption can be understood as a non-degeneracy hypothesis because two different pointer states α, β do not induce identical partial measurement results distribution $p^o(i|\alpha)$. Our aim is to generalize the convergence of $Q_n(\cdot)$ while weakening the hypotheses made in [1]. We discuss the convergence when different partial measurement methods are used. We focus on the influence of this extension on the rate of convergence. The degenerate case will be studied in section 5.1. In the case of one measurement method, a convergence result similar to that of [1] has been obtained by H. Amini, P. Rouchon and M. Mirrahimi through sub-martingale convergence in [3].

4.1 Convergence with different partial measurement methods

The extension of the convergence result of [1] to cases with different measurement methods is straightforward. From the fact that, conditional on \mathcal{F}_{n-1} ,

$$Q_n(\alpha) = \frac{Q_{n-1}(\alpha) p^{o_n}(i_n | \alpha)}{\sum_{\beta \in \mathcal{S}} Q_{n-1}(\beta) p^{o_n}(i_n | \beta)}$$

with probability $\sum_{\beta \in \mathcal{S}} Q_{n-1}(\beta) p^{o_n}(i_n | \beta)$, the average of $Q_n(\alpha)$, again conditioned on \mathcal{F}_{n-1} , is

$$\sum_{i_n \in \text{spec}_{o_n}} Q_n(\alpha) \sum_{\beta \in \mathcal{S}} Q_{n-1}(\beta) p^{o_n}(i_n | \beta) = Q_{n-1}(\alpha).$$

So $Q_{n-1}(\alpha)$ is conserved in average. Though the computation involved to prove it is essentially the same, a mathematically cleaner statement is that

$$\mathbb{E}(Q_n(\alpha)|\mathcal{F}_{n-1}) = Q_{n-1}(\alpha),$$

i.e. each $Q_n(\alpha)$ is an \mathbb{F} -martingale.

In fact, $Q_n(\alpha)$ has a deeper probabilistic meaning, which makes the martingale property obvious.

For a while, forget the previous definition of $Q_n(\alpha)$. Observe that, under the assumption that $q_0(\alpha) > 0$ for every $\alpha \in \mathcal{S}$, any set of \mathbb{P} -measure 0 has also \mathbb{P}_α -measure 0. Then the Radon-Nikodym theorem states that for each $\alpha \in \mathcal{S}$, there is a \mathbb{P} -integrable non-negative random variable $Q(\alpha)$ on (Ω, \mathcal{F}) such that

$$q_0(\alpha)\mathbb{E}_\alpha(X) = \mathbb{E}(Q(\alpha)X)$$

for every \mathbb{P}_α -integrable random variable X on (Ω, \mathcal{F}) . The random variable $Q(\alpha)$ is a Radon-Nikodym derivative of $q_0(\alpha)\mathbb{P}_\alpha$ with respect to \mathbb{P} . It is obvious that two Radon-Nikodym derivatives can differ only on a set of \mathbb{P} -measure 0: in that sense the Radon-Nikodym derivative is unique if it exists. We have also that, \mathbb{P} -almost surely, $\sum_\alpha Q(\alpha) = 1$, so that, \mathbb{P} -almost surely, each $Q(\alpha) \leq 1$. This existence theorem is a bit abstract but if one replaces \mathcal{F} by \mathcal{F}_n one can get a concrete formula. The same argument ensures the existence of a \mathbb{P} -integrable non-negative random variable $Q_n(\alpha)$ on (Ω, \mathcal{F}_n) such that $q_0(\alpha)\mathbb{E}_\alpha(X) = \mathbb{E}(Q_n(\alpha)X)$ for every \mathbb{P}_α -integrable random variable X on (Ω, \mathcal{F}_n) . As \mathcal{F}_n is finite, it suffices to let X run over the indicator functions for the $B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$. This implies that

$$Q_n(\alpha, \omega) = q_0(\alpha) \frac{\mathbb{P}_\alpha(B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}})}{\mathbb{P}(B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}})}$$

for every $\omega \in B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$ (such that the denominator is nonzero, else the value of $Q_n(\alpha, \omega)$ is immaterial). Explicitly one finds

$$Q_n(\alpha, \omega) = \frac{q_0(\alpha)p^{o_1}(i_1|\alpha) \cdots p^{o_n}(i_n|\alpha)}{\sum_{\beta \in \mathcal{S}} q_0(\beta)p^{o_1}(i_1|\beta) \cdots p^{o_n}(i_n|\beta)}$$

on $B_{o_1, i_1, o_2, \dots, i_n, o_{n+1}}$.

This is exactly our previous definition of $Q_n(\alpha)$, which is probabilistically a Radon-Nikodym derivative. This makes the martingale property obvious without any computation, just by the definition and general properties of conditional expectations. In fact, $Q_n(\alpha)$ is a closed martingale:

$$Q_n(\alpha) = \mathbb{E}(Q(\alpha)|\mathcal{F}_n).$$

As $Q_n(\alpha)$ is also bounded, the martingale convergence theorem [5] ensures that $Q_n(\alpha) \rightarrow Q(\alpha)$ \mathbb{P} -almost surely and in $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Since by assumption \mathcal{S} is a finite set, all $Q_n(\alpha)$'s converge simultaneously \mathbb{P} -almost surely.

Let us now observe that the following two statements are equivalent:

- the measures \mathbb{P}_α are all mutually singular,
- there is a collection $(\Omega_\alpha)_{\alpha \in \mathcal{S}}$ of disjoint measurable subsets of Ω such that, \mathbb{P} -almost surely $\mathbb{1}_{\Omega_\alpha} = Q(\alpha)$.

The proof is simple. The statement that the measures \mathbb{P}_α are all mutually singular is equivalent to the existence of a collection $(\Omega_\alpha)_{\alpha \in \mathcal{S}}$ of disjoint measurable subsets of Ω such that $\mathbb{P}_\beta(\Omega_\alpha) = \delta_{\alpha,\beta}$ for all $\alpha, \beta \in \mathcal{S}$. From the defining property of Radon-Nikodym derivatives, $\mathbb{1}_{\Omega_\alpha} Q(\alpha)$ is also a Radon-Nikodym derivative of $q_0(\alpha) \mathbb{P}_\alpha$ with respect to \mathbb{P} , and $\mathbb{E}(Q(\alpha) \mathbb{1}_{\Omega_\beta}) = q_0(\alpha) \delta_{\alpha,\beta}$ which by positivity implies that, for $\alpha \neq \beta$, $Q(\alpha) \mathbb{1}_{\Omega_\beta} = 0$ except maybe on a set of \mathbb{P} -measure 0. Hence \mathbb{P} -almost surely $\mathbb{1}_{\Omega_\alpha} Q(\beta) = Q(\beta) \delta_{\alpha,\beta}$. Summing over β gives $\mathbb{1}_{\Omega_\alpha} = Q(\alpha)$ \mathbb{P} -almost surely. The converse is also true: if there is a collection $(\Omega_\alpha)_{\alpha \in \mathcal{S}}$ of disjoint measurable subsets of Ω such that \mathbb{P} -almost surely $\mathbb{1}_{\Omega_\alpha} = Q(\alpha)$, then the measures \mathbb{P}_α are all mutually singular and concentrated on the Ω_α 's.

A striking consequence is that, if the measures \mathbb{P}_α are all mutually singular, for each ω in a set of \mathbb{P} -measure 1, $Q_n(\alpha)$ converges to either 0 or 1, and it converges to 1 with probability $\mathbb{P}(\Omega_\alpha) = q_0(\alpha)$.

Hence when the measures \mathbb{P}_α are all mutually singular *there is a full experimental equivalence between an infinite sequence of partial measurements and a direct projective measurement on the system*. We further study this equivalence in section 4.2.

We shall now give a criterion, that the experimenter may enforce on the protocol, ensuring that the measures \mathbb{P}_α are all mutually singular. This involves a non-degeneracy hypothesis, similar but weaker than that made in [1].

We say that $o \in \mathcal{O}$ is recurrent in $\omega \in \Omega$ if $O_n(\omega) = o$ for infinitely many n 's. Our (sufficient) criterion for all the measures \mathbb{P}_α to be mutually singular is:

There is a subset \mathcal{O}_s of \mathcal{O} such that

- Each $o \in \mathcal{O}_s$ is recurrent with probability 1 under each \mathbb{P}_α
- For every $\alpha, \beta \in \mathcal{S}$, $\alpha \neq \beta$ there is some $o \in \mathcal{O}_s$ and $i \in \text{spec}(o)$ such that $p^o(i|\alpha) \neq p^o(i|\beta)$.

This condition says that with probability one, infinitely many partial measurements that distinguish between any two states of the system will occur.

Consider the event A_o made of the ω 's such that $Q_n(\alpha)$ converges for each α and o is recurrent. Note that by our assumptions $\mathbb{P}(A_o) = 1$. We show that for any $i \in \text{spec}(o)$

$$Q(\alpha) \sum_{\gamma} Q(\gamma) p^o(i|\gamma) = Q(\alpha) p^o(i|\alpha)$$

on A_o . There are two cases to consider. Either $(O_n(\omega), I_n(\omega)) = (o, i)$ for infinitely many n 's: then the announced relation follows by taking the limit of the basic recursion relation along a subsequence. Or $(O_n(\omega), I_n(\omega)) = (o, i)$ for only finitely many n 's: then, as shown in Appendix A, $\sum_\gamma Q(\gamma)p^o(i|\gamma) = 0$ so that in particular $Q(\alpha)p^o(i|\alpha) = 0$ and the announced relation still holds.

This implies that

$$\forall \alpha, \beta \in \mathcal{S}, Q(\alpha)Q(\beta)(p^o(i|\alpha) - p^o(i|\beta)) = 0 \text{ on } A_o \text{ for every } i \in \text{spec}(o). \quad (7)$$

Then, by (7), $\forall \alpha, \beta \in \mathcal{S}, \alpha \neq \beta$ one has $Q(\alpha)Q(\beta) = 0$ on $A_{\mathcal{O}_s} := \cap_{o \in \mathcal{O}_s} A_o$, which has \mathbb{P} -measure 1. As the sum of the $Q(\alpha)$'s is 1, this means that on $A_{\mathcal{O}_s}$ one has $Q(\alpha) = \delta_{\alpha, \Upsilon}$ where $\Upsilon(\omega)$ is some ω -dependent element of \mathcal{S} . So there is a family of disjoint subsets Ω_α of $A_{\mathcal{O}_s}$ such that $\cup_\alpha \Omega_\alpha$ has \mathbb{P} -measure 1, and $Q(\alpha) = \mathbb{1}_{\Omega_\alpha}$ except maybe on a set of \mathbb{P} -measure 0.

We shall give two examples.

For the first one, the task to ensure that enough measurement methods o are recurrent is left to the experimenter.

The second example is the Markovian feedback protocol. For this protocol, under \mathbb{P}_α , the process (o_n, i_n) is Markovian with transition kernel $K_{\alpha, n}(o, i; o', i') = p^{o'}(i'|\alpha)c_n(o'|o, i)$, with initial distribution $p^o(i|\alpha)c_0(o)$. Recurrence questions are well under control at least when the kernels do not depend on n . So we assume that $c_n(o'|o, i) = c(o'|o, i)$ is time independent² and set $K_\alpha(o, i; o', i') = p^{o'}(i'|\alpha)c(o'|o, i)$. The product structure of K_α leads to introduce the reduced transition kernel $K_\alpha^{red}(o; o') := \sum_{i \in \text{spec}(o)} p^o(i|\alpha)c(o'|o, i)$. One can rely on classical Markov chain computations to make sure that the measures \mathbb{P}_α are all mutually singular. Assuming that the reduced Markov chain K_α^{red} is irreducible and aperiodic, it admits a unique invariant probability μ_α^{red} on \mathcal{O} , which is strictly positive. Then all partial measurement methods with $\mu_\alpha^{red}(o) > 0$ will be recurrent on a set of \mathbb{P}_α -measure 1. Moreover the full Markov chain has a unique invariant probability $\mu_\alpha(o, i) = p^o(i|\alpha)\mu_\alpha^{red}(o)$. Then the strong law of large numbers for Markov chains states that $N_n(o, i)$, the number of occurrence of (o, i) up to the n^{th} experiment, satisfies

$$N_n(o, i) \sim n\mu_\alpha(o, i) \text{ for large } n$$

on a set of \mathbb{P}_α -measure 1. This ergodicity result will be put in use in section 4.3.

To summarize, we have proved that if there are enough \mathbb{P}_α recurrent partial measurement methods then the measures \mathbb{P}_α are all mutually singular,

²Were c_n periodic in n we could look at a Markov chain with a larger state space to reduce to that case.

so that there is a full experimental equivalence between an infinite sequence of partial measurements and a direct projective measurement on the system.

4.2 Conditioning or projecting

In the previous section we pointed out the connection between complete measurements and direct projective measurements. This holds whenever the measures \mathbb{P}_α are mutually singular.

Under this hypothesis, we show in this section that we can solve the random recursion relation (3) on a space where the final pointer state is determined before the measurement process starts: for the class of experiments we are dealing with, it is consistent to assess that the total measurement outcome can be decided in advance and by a classical probabilistic choice.

We assume that the measures \mathbb{P}_α are mutually singular, and, as usual, that all $q_0(\alpha)$ are > 0 . To avoid clumsy statements, we remove from Ω the (\mathbb{P} -negligible) set of events for which either $Q_n(\alpha)$ is not defined for all n 's, or $Q_n(\alpha)$ does not converge to 0 or 1. So we assume that the sets Ω_α form a partition of Ω , and the random variable Υ defined by $\lim_{n \rightarrow \infty} Q_n(\alpha, \omega) = \mathbb{1}_{\omega \in \Omega_\alpha} = \delta_{\alpha, \Upsilon(\omega)}$ is defined everywhere on Ω . We let \mathcal{A} be the smallest σ -algebra making any Ω_α measurable. We claim that if $X \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ is any integrable random variable,

$$\mathbb{E}(X|\mathcal{A}) = \mathbb{E}_\Upsilon(X).$$

This can be rephrased as : conditioning \mathbb{P} on the limit of $Q_n(\alpha)$ being 1 leads to \mathbb{P}_α . This is essentially obvious from the Radon-Nikodym viewpoint and the fact that the measures \mathbb{P}_α are mutually singular. But a direct computation is easy. The fact that \mathcal{S} is finite (countable would do the job as well) has two consequences. First, any \mathcal{A} -measurable random variable Y can be written as a linear combination $Y = \sum_{\alpha \in \mathcal{S}} y_\alpha \mathbb{1}_{\Omega_\alpha}$. Second, to test that $\mathbb{E}(X|\mathcal{A}) = Y$ it suffices to check that $\mathbb{E}(X \mathbb{1}_{\Omega_\alpha}) = \mathbb{E}(Y \mathbb{1}_{\Omega_\alpha})$ for every α . Now, by definition, $\mathbb{E}(X \mathbb{1}_{\Omega_\alpha}) = q_0(\alpha) \mathbb{E}_\alpha(X)$, whereas $\mathbb{E}(Y \mathbb{1}_{\Omega_\alpha}) = q_0(\alpha) y_\alpha$, so, if $\mathbb{E}(X|\mathcal{A}) = Y$, $y_\alpha = \mathbb{E}_\alpha(X)$, i.e. $Y = \sum_{\alpha \in \mathcal{S}} \mathbb{E}_\alpha(X) \mathbb{1}_{\Omega_\alpha} = \mathbb{E}_\Upsilon(X)$. Hence $\mathbb{E}(X|\mathcal{A}) = \mathbb{E}_\Upsilon(X)$ as announced.

This proves the equivalence between projecting first on a given state α and conditioning on the limit state being α .

4.3 Convergence rates and trial distribution independence

Experimentally, the initial distribution $q_0(\cdot)$ may not be known. One would then use the sequence of partial measurements to gain information and reconstruct it from these measurements. This may be done using Bayes' law starting from a trial distribution $\hat{q}_0(\cdot)$ (supposed to be nowhere vanishing)

and recursively improving it using the relation

$$\hat{q}_n(\alpha) = \hat{q}_{n-1}(\alpha) \frac{p^{o_n}(i_n|\alpha)}{\sum_{\beta} \hat{q}_{n-1}(\beta) p^{o_n}(i_n|\beta)}$$

if the outcome is i , which happens with probability $\sum_{\beta} q_{n-1}(\beta) p^{o_n}(i_n|\beta)$. The difference with eq.(3) is that the recursion involves $\hat{q}_n(\cdot)$ and not $q_n(\cdot)$. However, the probability is the one given by the $q_n(\cdot)$. If the initial trial distribution $\hat{q}_0(\cdot)$ coincides with the initial system distribution $q_0(\cdot)$, then $\hat{q}_n(\cdot) = q_n(\cdot)$ for all n . Both $\hat{q}_n(\cdot)$ and $q_n(\cdot)$ are realization dependent. We shall define the random process $\hat{Q}_n(\cdot)$ as $Q_n(\cdot)$ in (6) but with a different initial distribution

$$\hat{Q}_n(\alpha) := \hat{q}_0(\alpha) \frac{p^{o_1}(i_1|\alpha) \cdots p^{o_n}(i_n|\alpha)}{\sum_{\beta} \hat{q}_0(\beta) p^{o_1}(i_1|\beta) \cdots p^{o_n}(i_n|\beta)}$$

The probability law still depends on the true initial distribution. Notice that $\hat{Q}_n(\cdot)$ is not an \mathbb{F} -martingale under this law, contrary to $Q_n(\cdot)$. As we shall show, they nevertheless have identical limit, that is: $\lim_{n \rightarrow \infty} \hat{Q}_n(\alpha)$ exists and is equal to $\mathbb{1}_{\Omega_{\alpha}}$ with \mathbb{P} -probability 1.

Moreover, if a time independent Markovian feedback protocol is used, the convergence of the state probability distribution is exponential. Its convergence rate is the mean relative entropy of the partial measurement result distribution conditioned on the system be in the state Υ with respect to the one conditioned on the system be in the state α . This means that for n large enough,

$$\hat{Q}_n(\alpha) \simeq e^{-n\bar{S}(\Upsilon|\alpha)}, \quad \text{for } \alpha \neq \Upsilon \quad (8)$$

with

$$\bar{S}(\beta|\alpha) := \sum_{o \in \mathcal{O}} \mu_{\Upsilon}^{red}(o) \sum_{i \in \text{spec}(o)} p^o(i|\beta) \ln \left[\frac{p^o(i|\beta)}{p^o(i|\alpha)} \right] \quad (9)$$

Here, all the $p^o(i|\alpha)$ are assumed to be strictly positive, thus any (o, i) with $o \in \mathcal{O}_s$ is recurrent. In the case of time independent random protocols, the rate is the same with $\mu_{\Upsilon}^{red}(\cdot)$ replaced by $c(\cdot)$ the distribution of measurement methods. This coincides with the result of [1] if \mathcal{O} contains only one partial measurement method.

The independence of the limiting distribution with respect to the initial trial distribution is obtained whenever one starts with a trial distribution such that $\hat{q}_0(\alpha) > 0$ wherever $q_0(\alpha) > 0$. This happens for example if we start with $\hat{q}_0(\alpha) > 0$ for any $\alpha \in \mathcal{S}$.

To see this, we analyse the behavior of \hat{Q}_n under the probability measure $\hat{\mathbb{P}} := \sum_{\alpha \in \mathcal{S}} \hat{q}_0(\alpha) \mathbb{P}_{\alpha}$, which can be seen as a trial probability measure on Ω

as it corresponds to a system initially in the trial state. Under $\hat{\mathbb{P}}$, $\hat{Q}_n(\alpha)$ is a martingale, so by the above arguments, it converges $\hat{\mathbb{P}}$ almost surely to $\mathbb{1}_{\Omega_\alpha}$. As by hypothesis $\hat{q}_0(\alpha) > 0$ whenever $q_0(\alpha) > 0$ (which can be rephrased as: \mathbb{P} is absolutely continuous with respect to $\hat{\mathbb{P}}$), a subset of Ω of $\hat{\mathbb{P}}$ probability 1 has also \mathbb{P} probability 1. So $\lim_{n \rightarrow \infty} \hat{Q}_n(\alpha) = \mathbb{1}_{\Omega_\alpha} = \lim_{n \rightarrow \infty} Q_n(\alpha)$ with \mathbb{P} probability 1.

What is less direct is the determination of the convergence rate. This requires controlling the behavior of the counting processes $N_n(o, i)$. As recalled at the end of section 4.1, $N_n(o, i)/n \rightarrow \mu_\alpha(o, i)$ on a set³ of \mathbb{P}_α -measure 1. We want to infer that \mathbb{P} -almost surely,

$$\lim_{n \rightarrow \infty} N_n(o, i)/n = \mu_\Upsilon(o, i).$$

To prove it, we set $\mathcal{L}(o, i) := \{\lim_{n \rightarrow \infty} N_n(o, i)/n = \mu_\Upsilon(o, i)\}$ and write

$$\mathbb{P}(\mathcal{L}(o, i)) = \sum_{\alpha} q_0(\alpha) \mathbb{P}(\mathcal{L}(o, i) | \Omega_\alpha) = \sum_{\alpha} q_0(\alpha) = 1,$$

where we used

$$\begin{aligned} \mathbb{P}(\mathcal{L}(o, i) | \Omega_\alpha) &= \mathbb{P}(\lim_{n \rightarrow \infty} N_n(o, i)/n = \mu_\alpha(o, i) | \Omega_\alpha) \\ &= \mathbb{P}_\alpha(\lim_{n \rightarrow \infty} N_n(o, i)/n = \mu_\alpha(o, i)) = 1. \end{aligned}$$

Observe now that $\hat{Q}_n(\alpha)$ can be expressed as a function of the counting processes:

$$\hat{Q}_n(\alpha) = \frac{\hat{q}_0(\alpha) \prod_{(o, i) \in E} p^o(i | \alpha)^{N_n(o, i)}}{\sum_{\beta} \hat{q}_0(\beta) \prod_{(o, i) \in E} p^o(i | \beta)^{N_n(o, i)}}$$

Under the hypothesis that all $p^o(i | \alpha)$'s are > 0 , we have that $\hat{Q}_n(\alpha) > 0$ for every n , and the logarithm of the ratio between $\hat{Q}_n(\alpha)$ and $\hat{Q}_n(\beta)$ is well defined. Using the previous result, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left[\frac{\hat{Q}_n(\beta)}{\hat{Q}_n(\alpha)} \right] &= \sum_{(o, i) \in E} \mu_\Upsilon(o, i) \ln \left[\frac{p^o(i | \beta)}{p^o(i | \alpha)} \right] \\ &= \sum_{(o, i) \in E} \mu_\Upsilon^{red}(o) p^o(i | \Upsilon) \left(\ln \left[\frac{p^o(i | \beta)}{p^o(i | \Upsilon)} \right] + \ln \left[\frac{p^o(i | \Upsilon)}{p^o(i | \alpha)} \right] \right) \end{aligned}$$

Then, for a large enough n ,

$$\frac{\hat{Q}_n(\beta)}{\hat{Q}_n(\alpha)} \simeq e^{-n \bar{S}(\Upsilon | \beta)} e^{n \bar{S}(\Upsilon | \alpha)}$$

³For the record, the set $\{N_n(o, i)/n \rightarrow \mu_\alpha(o, i)\}$ is measurable, as it can be written $\bigcap_{m \in \mathbb{N}^*} \bigcup_{n_0 \in \mathbb{N}} \bigcap_{n > n_0} \{\omega \in \Omega, |N_n(o, i; \omega)/n - \mu_\alpha(o, i)| < 1/m\}$.

with $\bar{S}(\Upsilon|\alpha)$ the mean relative entropy, $\bar{S}(\Upsilon|\alpha) = \sum_{o \in \mathcal{O}} \mu_{\Upsilon}^{red}(o) S^o(\Upsilon|\alpha)$ where

$$S^o(\Upsilon|\alpha) = \sum_{i \in \text{spec}(o)} p^o(i|\Upsilon) (\ln[p^o(i|\Upsilon)] - \ln[p^o(i|\alpha)])$$

The relative entropy is always non negative, subsequently, the mean relative entropy is non negative too. Moreover, the mean relative entropy is null if and only if $\Upsilon = \beta$ (all relative entropies null).

Using this property and $\sum_{\beta} \hat{Q}_n(\beta) = 1$, we obtain for $\alpha \neq \Upsilon$,

$$\hat{Q}_n(\alpha)^{-1} = \sum_{\beta} \frac{\hat{Q}_n(\beta)}{\hat{Q}_n(\alpha)} \simeq e^{n\bar{S}(\Upsilon|\alpha)} (1 + \sum_{\beta \neq \Upsilon} e^{-n\bar{S}(\Upsilon|\beta)})$$

Then, to leading exponential order

$$\hat{Q}_n(\alpha) \simeq e^{-n\bar{S}(\Upsilon|\alpha)}$$

Hence, for n large enough, we proved that

$$\hat{Q}_n(\alpha) \simeq \begin{cases} 1 & \text{if } \alpha = \Upsilon \\ \text{const.} e^{-n\bar{S}(\Upsilon|\alpha)} & \text{else} \end{cases}$$

The limit distribution does not depend on the trial initial distribution but only on the complete measurement realization. The probability to have $\hat{Q}_{\infty}(\alpha) = \delta_{\alpha,\gamma}$ equals $q_0(\gamma)$. With time independent Markovian feedback protocol, the convergence is exponential with a leading rate $\bar{S}(\Upsilon|\alpha)$.

4.4 Convergence rate tuning

Most of the time, when performing a measurement, one prefers it to take as little time as possible. The use of different partial measurement methods allows us to tune the convergence rate. Let us take an example. Suppose we want to discriminate between three possible pointer states of a system, and suppose that the partial measurement methods give only **True/False** as possible outputs. We denote T, F the partial measurement results and $1, 2, 3$ the pointer states. Each partial measurement method can be tuned to maximize, up to a measurement error $\varepsilon \ll 1$, the probability of one of its outcome knowing the system is in one of the three pointer states. We shall show that this is not enough to maximize all convergence rates for arbitrary limit pointer state. It is the use of different measurement methods picked randomly that allows us to overcome this convergence rate problem.

Let us consider for instance two measurement methods. The first one, denoted a , has conditioned probabilities

$$p^a(T|1) = \varepsilon, p^a(T|2) = q, p^a(T|3) = 1 - \varepsilon$$

with $q = O(1)$. The second one, denoted b , is obtained by switching the probability conditioned on 1 and 2, that is

$$p^b(T|1) = q, p^b(T|2) = \varepsilon, p^b(T|3) = 1 - \varepsilon$$

Let us now look at the convergence rate conditioned on the limit pointer state to be 1. These are coded in the relative entropies. If only the measurement method a is used, one has:

$$\begin{aligned} S^a(1|2) &= \varepsilon \ln \left[\frac{\varepsilon}{q} \right] + (1 - \varepsilon) \ln \left[\frac{1 - \varepsilon}{1 - q} \right] \sim -\ln[1 - q] = O(1) \\ S^a(1|3) &= \varepsilon \ln \left[\frac{\varepsilon}{1 - \varepsilon} \right] + (1 - \varepsilon) \ln \left[\frac{1 - \varepsilon}{\varepsilon} \right] \sim -\ln[\varepsilon] \gg 1 \end{aligned}$$

The convergence of $Q_n(3)$ toward 0 is quick but the one of $Q_n(2)$ is rather slow. If measurement method b is used the interesting relative entropies are now

$$\begin{aligned} S^b(1|2) &= q \ln \left[\frac{q}{\varepsilon} \right] + (1 - q) \ln \left[\frac{1 - q}{1 - \varepsilon} \right] \sim -\ln[\varepsilon] \\ S^b(1|3) &= q \ln \left[\frac{q}{1 - \varepsilon} \right] + (1 - q) \ln \left[\frac{1 - q}{\varepsilon} \right] \sim -\ln[\varepsilon] \end{aligned}$$

All the convergences rates are then high if the limit pointer state is 1. But if the limit pointer state is not 1 but 2 then, using only the measurement method b , the relative entropy $S^b(2|1)$ is

$$S^b(2|1) = \varepsilon \ln \left[\frac{\varepsilon}{q} \right] + (1 - \varepsilon) \ln \left[\frac{1 - \varepsilon}{1 - q} \right] \sim -\ln[q] = O(1)$$

and the convergence rate toward 2 is slow.

Now, if at each time one of the two measurement methods is used with equal probability $\frac{1}{2}$. The convergence rate for any i, j with $i \neq j$ is

$$\overline{S}(i|j) = \frac{1}{2}(S^a(i|j) + S^b(i|j)) \sim -\ln[\varepsilon] \gg 1$$

As a consequence, the convergence rate is always high, whichever the limit pointer state is.

In the toy model, if the first partial measurement method correspond to $\theta - \theta' = \frac{\pi}{3}$, then

$$S^{\frac{\pi}{3}}(0|3) \sim 0.116$$

This is the slowest of all convergence rates. If the partial measurement method with $\theta - \theta' = \frac{\pi}{6}$ is introduced and the partial measurement methods are chosen with equal probabilities each time, then

$$\overline{S}(0|3) \sim 1.18$$

and the slowest of all convergence rates is

$$\overline{S}(1|3) \sim 1.10$$

If only the $\frac{\pi}{3}$ measure is used and the limit pointer state is 0, then a theoretical 99% confidence level is reached after about 50 measures. With the use of the two different partial measurement methods the same confidence level for the same limit state is reached in 5 measures. The same number of measurements is needed if the limit pointer state is 1, 2 or 3.

5 Degeneracy and limit quantum state

Often the quantity we measure is a property common to several pointer states. In the quantum case, this corresponds to a degenerate projective Von Neumann measurement. There, at least two different eigenstates share the same eigenvalue. For our measurement process, degeneracies happen when several distributions $p^o(\cdot)$ are equal for different pointer states, so that some states cannot be distinguished. For example, in our toy model, whatever $\theta - \theta'$ is, we have $p^{\theta-\theta'}(\pm|p) = p^{\theta-\theta'}(\pm|p + 4k)$ with k an integer. The pointer state with p photons cannot be distinguished from the one with $p + 4k$ photons.

In this section we study the system state evolution when degenerate repeated partial measurements are performed. In a first part we show the convergence of the system pointer state distribution. In a second part we focus on the quantum case and the influence of phases introduced between degenerate states by the repeated partial measurement process.

We shall partition the set of configurations into sectors. Let us define an equivalence relation among pointers by identifying two pointers whose partial measurement distributions are identical. That is: two pointers α and β are said to be equivalent (denoted $\alpha \sim \beta$) if, for any partial measurement method o and result i ,

$$p^o(i|\alpha) = p^o(i|\beta).$$

By definition the sector $\underline{\alpha}$ is the equivalence class of α . In the toy model the sectors are the sets $\underline{p} = \{p + 4k, k \in \mathbb{N}\}$ with $p = 0, 1, 2, 3$.

5.1 State distribution convergence

We first look at the convergence of the pointer state distribution $Q_n(\cdot)$ in case of degeneracy. The system distributions $Q_n(\cdot)$ induce probability distributions $\bar{Q}_n(\cdot)$ on sectors by

$$\bar{Q}_n(\underline{\alpha}) := \sum_{\alpha' \in \underline{\alpha}} Q_n(\alpha').$$

Since sectors form a partition of the set of pointer states, we have $\sum_{\underline{\alpha}} Q_n(\underline{\alpha}) = 1$. The initial probability of a sector is $\bar{q}_0(\underline{\alpha}) = \sum_{\alpha' \in \underline{\alpha}} q_0(\alpha')$. The recursion relation (3) can obviously be lifted to a recursion relation for the sector distributions,

$$\bar{Q}_{n+1}(\underline{\alpha}) = \bar{Q}_n(\underline{\alpha}) \frac{p^{o_n}(i_n|\alpha)}{\sum_{\underline{\beta}} \bar{Q}_n(\underline{\beta}) p^{o_n}(i_n|\beta)}.$$

It is identical in structure to eq.(3) but with the bonus that it now is non degenerate. Lets assume that for two different sectors, it exists at least one \mathbb{P} -recurrent partial measurement method distinguishing between the two sectors: if $\alpha \not\sim \beta$, it exists $o \in \mathcal{O}_s$ and $i \in \text{spec}(o)$ such that $p^o(i|\alpha) \neq p^o(i|\beta)$. Thus we can use the non-degenerate case results but applied to the sector distribution. Hence, $\bar{Q}_n(\cdot)$ almost surely converge and

$$\bar{Q}(\underline{\alpha}) = \delta_{\underline{\alpha}, \underline{\Upsilon}}$$

with $\underline{\Upsilon}$ the realization dependent limit sector. The probability that the limit sector be $\underline{\gamma}$ is equal to $\bar{q}_0(\underline{\gamma})$.

From the martingale property, the state distribution converge (not only the sector distribution), and the point which remains to be discussed is what is this limit. Thanks to the relation $q_0(\alpha') Q_n(\alpha) = q_0(\alpha) Q_n(\alpha')$ valid for any n if $\alpha \sim \alpha'$, we shall show that this limit is

$$Q(\alpha) = \begin{cases} 0 & \text{if } \alpha \notin \underline{\Upsilon} \\ q_0(\alpha)/\bar{q}_0(\underline{\Upsilon}) & \text{if } \alpha \in \underline{\Upsilon} \end{cases} \quad (10)$$

with $\underline{\Upsilon}$ the limit sector.

Indeed, the state distribution satisfies the recursion relation (3). Thus, if $(i_k)_{k=0, \dots, n-1}$ are the n first partial measurements results, one has

$$\frac{Q_n(\alpha')}{q_0(\alpha')} = \frac{\prod_{k=1}^n p^{o_k}(i_k|\alpha)}{\sum_{\beta} q_0(\beta) \prod_{k=1}^n p^{o_k}(i_k|\beta)}$$

for any α' in the sector $\underline{\alpha}$. The right hand side only depends on the sector $\underline{\alpha}$, and thus $\frac{Q_n(\alpha)}{q_0(\alpha)} = \frac{Q_n(\alpha')}{q_0(\alpha')}$ if $\alpha' \in \underline{\alpha}$. From this equality, it follows that

$$Q(\alpha') = Q(\alpha) \frac{q_0(\alpha')}{q_0(\alpha)}$$

Since, $\bar{Q}(\underline{\alpha}) = \delta_{\underline{\alpha}, \underline{\Upsilon}}$, we have $Q(\alpha) = 0$ if $\alpha \notin \underline{\Upsilon}$ and $1 = \sum_{\alpha' \in \underline{\Upsilon}} Q(\alpha') = \frac{Q(\alpha)}{q_0(\alpha)} \bar{q}_0(\underline{\Upsilon})$, for any $\alpha \in \underline{\Upsilon}$. Hence,

$$Q(\alpha) = \frac{q_0(\alpha)}{\bar{q}_0(\underline{\Upsilon})}, \quad \text{for } \alpha \in \underline{\Upsilon}$$

The probability of convergence to a sector as well as the limit state distribution (10) coincide, in quantum mechanics, with what would have been

predicted by Von Neumann rules for degenerate projective measurements. The approach we have been following so far, based on tools from classical probability theory, gives no information on the convergence of the density matrix off-diagonal elements. It is the next section's purpose to discuss the evolution of the density matrix ρ_n and not only the evolution of the probabilities $Q_n(\alpha) = \langle \alpha | \rho_n | \alpha \rangle$.

5.2 Density matrix convergence

We are now interested in the convergence of the system density matrix. In the basis of pointer states we may write:

$$\rho_n = \sum_{\alpha, \beta} A_n(\alpha, \beta) |\alpha\rangle \langle \beta|$$

with $A_n(\alpha, \alpha) = Q_n(\alpha)$. It evolves according to the recursion relation (1). The processes $A_n(\alpha, \beta)$, are not martingales, and their convergence can not be obtained through the martingale convergence theorem. Actually, they do not always converge. To obtain convergence a unitary evolution process has to be subtracted.

For each POVM, a phase between the pointer states is introduced by the operators $M_i^{(o)}$. Even inside a sector this phase can be nonzero. This possibility comes from the degeneracy criteria we unraveled previously. Two pointer states, α, β can have a nonzero limit probability if they are in the same sector : $\alpha \sim \beta$. This criterion implies a norm equality $|M^{(o)}(i|\alpha)| = |M^{(o)}(i|\beta)|$ for any i in the spectrum of any partial measurement, but not a full equality. So $M^{(o)}(i|\alpha)$ and $M^{(o)}(i|\beta)$ can differ by a phase. The density matrix converges either if this phase can be set to zero or if we absorb it through a transformation of the evolution.

Let us write the operators $M^{(o)}(i|\alpha)$ in a phase times norm form

$$M^{(o)}(i|\alpha) = e^{-i\Delta t(E_\alpha + \theta^{(o)}(i|\alpha))} \sqrt{p^{(o)}(i|\alpha)}$$

The specific form of the phase is inspired by the Hamiltonian $H^{(o)} = \sum_\alpha (E_\alpha \mathbb{I} + H_p^{(o)} + H_\alpha^{(o)}) |\alpha\rangle \langle \alpha|$. This is the most general Hamiltonian if one wants U to fulfill the non demolition condition (2). In the above formula, Δt is the interaction time between the probe and the system.

Let us define a unitary operator process, diagonal in the pointer state basis,

$$\tilde{U}_n = \sum_\alpha e^{-i\Delta t(nE_\alpha + \sum_{(o,i) \in E} \theta^{(o)}(i|\alpha) N_n(o,i))} |\alpha\rangle \langle \alpha|$$

and the unitary equivalent conjugate density matrix process

$$\tilde{\rho}_n = \tilde{U}_n^\dagger \rho_n \tilde{U}_n \tag{11}$$

The diagonal elements of ρ_n in the basis $\{|\alpha\rangle\}$, are not affected by this transformation. Their limits stay the same. Thus if α or β are not in the limit sector $\underline{\Upsilon}$, according to the Cauchy-Schwartz theorem, $\tilde{A}_\infty(\alpha, \beta) = 0$. We are then interested in the limit of the elements $\tilde{A}_n(\alpha, \beta)$ with $\alpha, \beta \in \underline{\Upsilon}$.

If $\beta \in \underline{\alpha}$, then $q_0(\alpha)\tilde{A}_n(\alpha, \beta) = a_0(\alpha, \beta)Q_n(\alpha)$. Repeating the discussion made in the section 5.1, we get

$$\tilde{A}_\infty(\alpha, \beta) = \begin{cases} a_0(\alpha, \beta)/\bar{q}_0(\underline{\Upsilon}) & \text{if } \alpha, \beta \in \underline{\Upsilon} \\ 0 & \text{else} \end{cases}$$

Hence, $\tilde{\rho}_n$ has an almost sure limit which coincides with the result of a Von Neumann measurement: $\tilde{\rho}_\infty$ is equal to ρ_0 projected on the system subspace corresponding to the sector $\underline{\Upsilon}$.

$$\lim_{n \rightarrow \infty} \tilde{\rho}_n = \frac{1}{\bar{q}_0(\underline{\Upsilon})} P_{\underline{\Upsilon}} \rho_0 P_{\underline{\Upsilon}} \quad (12)$$

where $P_{\underline{\Upsilon}} := \sum_{\gamma \in \underline{\Upsilon}} |\gamma\rangle\langle\gamma|$ is the projector on the subspace corresponding to the sector $\underline{\Upsilon}$.

In some cases the unitary operator process can be reduced to a deterministic one. For example if $\theta^{(o)}(i|\alpha) = \theta^{(o)}(i|\beta)$ for every $(o, i) \in E$ and every $\alpha \sim \beta$, then we can chose $\tilde{U}_n = e^{-i\Delta t H_s}$. This is the case in the toy model, from the periodicity of trigonometric functions, in a given sector all phases introduced by partial measurements are equal. If the limit sector is $\underline{p} = \{p + 4k, k \in \mathbb{N}\}$, then the photon field limit state will be

$$\lim_{n \rightarrow \infty} e^{in\Delta t \hat{p}} \rho_n e^{-in\Delta t \hat{p}} = \sum_{k, k' \in \mathbb{N}} \frac{a_0(p + 4k, p + 4k')}{\bar{q}_0(\underline{p})} |p + 4k\rangle\langle p + 4k'|$$

One other example corresponds to the case where the phases $\theta^{(o)}(i|\alpha)$ do not depend on (o, i) but depend on α . Then one can define $H_{eff.} = H_s + \sum_{\alpha} \theta(\alpha) |\alpha\rangle\langle\alpha|$ and $\tilde{U}_n = e^{-in\Delta t H_{eff.}}$.

In most of the cases the unitary evolution \tilde{U}_n is a stochastic process and then in the limit $n \rightarrow \infty$, it remains a stochastic rotation inside the limit sector. When \tilde{U}_n is deterministic, the remaining rotation is deterministic too.

6 Continuous diffusive limit

We shall now prove the convergence of the discrete processes we consider toward processes driven by time continuous Belavkin diffusive equations. Our proof, different from that used in [21], allows us to derive the continuous measurement diffusive equation not only for the quantum repeated indirect measurement process but also for the macroscopic Bayesian apparatus we defined. The quantum case is a peculiar realization of it.

The time continuous equation is found as a scaling limit of the discrete evolution when n goes to infinity with $t = n\delta$ fixed ($\delta = \Delta t$). We first study the pointer state distribution scaling limit, $Q_t(\alpha) := \lim_{\delta \rightarrow 0} Q_{[t/\delta]}(\alpha)$. The evolution equation for $Q_t(\alpha)$ is given in eq.(15) below. We then look at the time continuous limit of the density matrix evolution and get the Belavkin diffusive equation eq.(23). In the quantum case, the continuous limit requires rescaling appropriately the system-probe interaction Hamiltonian as

$$H = H_s \otimes \mathbb{I}_p + \mathbb{I}_s \otimes H_p + \frac{1}{\sqrt{\delta}} H_I \quad (13)$$

We present in some details the case with a unique partial measurement method. The results are then easily extended to cases with different measurement methods.

6.1 Continuous time limit of the pointer state distribution

We are here interested in the state distribution continuous time limit. The results presented in this section apply to the general Bayesian recursion relation (3) — which in particular includes the case of repeated QND measurements. To begin with, we assume that there is only one partial measurement method⁴. Henceforth we suppress o from all the notations and let \mathcal{I} stand for the index set of outcomes. Note that the two filtrations \mathcal{F}_n and \mathcal{F}'_n coincide and carry the information on the first n partial measurements.

We assume that the conditional probabilities $p(i|\alpha)$ depend on a further small parameter δ , and are of the form

$$p(i|\alpha) = p_0(i)(1 + \sqrt{\delta} \Gamma_\delta(i|\alpha)) \quad (14)$$

with $p_0(i) > 0$ for all i 's and that $\Gamma(i|\alpha) := \lim_{\delta \rightarrow 0+} \Gamma_\delta(i|\alpha)$ exists. Then $\sum_i p_0(i) = 1$, so that the $p_0(i)$'s specify a probability measure, and for every δ , $\sum_i p_0(i) \Gamma_\delta(i|\alpha) = 0$. The important point is that $p_0(i)$ is independent of α .

These hypothesis are of course satisfied in the quantum case with QND interaction Hamiltonian $H_I = \sum_\alpha |\alpha\rangle\langle\alpha| \otimes H_\alpha$ and rescaling $H_I \rightarrow \frac{1}{\sqrt{\delta}} H_I$. Then $p_0(i) = |\langle i|\Psi\rangle|^2$ is the probability measure in absence of interaction, and

$$\Gamma(i|\alpha) := 2 \operatorname{Im} \left(\frac{\langle i|H_\alpha|\Psi\rangle}{\langle i|\Psi\rangle} \right)$$

We assume that $\langle i|\Psi\rangle \neq 0$ for all i .

We first need to make precise the sense in which a limit on $Q_n(\alpha)$ is to be taken.

⁴This condition will be relaxed in section 6.1.4.

For a fixed $\omega \in \Omega$ the limit $\lim_{\delta \rightarrow 0} Q_{[t/\delta]}(\alpha)$ is not expected to exist⁵. But there is some hope that, properly defined, a limit for the law of the process $Q_{[t/\delta]}(\alpha)$, $t \in \mathbb{R}^+$ exists. We shall prove in Appendix B that this is the case, specifying a bit the kind of convergence that is involved. Under the limiting law, the process $Q_t(\cdot)$ satisfies the stochastic equation

$$dQ_t(\alpha) = Q_t(\alpha) \sum_i (\Gamma(i|\alpha) - \langle \Gamma_i \rangle_t) dX_t(i) \quad (15)$$

where

$$\langle \Gamma_i \rangle_t := \sum_{\beta} Q_t(\beta) \Gamma(i|\beta).$$

Here $X_t(i)$, with $\sum_i X_t(i) = 0$, are continuous martingales with quadratic covariation

$$dX_t(i)dX_t(j) = dt (\delta_{i,j} p_0(i) - p_0(i)p_0(j)). \quad (16)$$

We shall show that a vector solving this equation is a bounded martingale, to which the martingale convergence theorem can be applied with results similar to those in the discrete case:

$$Q_{\infty}(\alpha) = \begin{cases} 0 & \text{if } \alpha \notin \underline{\Upsilon} \\ q_0(\alpha)/\bar{q}_0(\underline{\alpha}) & \text{if } \alpha \in \underline{\Upsilon} \end{cases}$$

with $\underline{\Upsilon}$ the limit sector. However, the sector definition is not the same as in the discrete case. In time continuous, α and β are in the same sector if and only if $\Gamma(i|\alpha) = \Gamma(i|\beta)$ for all partial measurement result i . The probability for the system to be in the sector $\underline{\alpha}$ in the limit t goes to infinity is $\bar{q}_0(\underline{\alpha}) = \sum_{\alpha' \in \underline{\alpha}} q_0(\alpha')$.

The convergence is still exponential

$$Q_t(\alpha) = \exp(-t/\tau_{\Upsilon\alpha}), \quad \text{if } \alpha \notin \underline{\Upsilon}$$

with characteristic convergence time $\tau_{\gamma\alpha}$,

$$2/\tau_{\gamma\alpha} = \sum_i p_0(i) (\Gamma(i|\alpha) - \Gamma(i|\gamma))^2 \quad (17)$$

This coincides with the convergence rate we would have found by taking the relative entropy $S(\gamma|\alpha)$ scaling limit. However, it is somewhat difficult to decipher that it originates from a relative entropy by only knowing its expression in the continuous-time limit.

⁵Think of the simple random walk: the convergence to Brownian motion is not sample by sample because $\frac{S_{2n}}{\sqrt{2n}}$ has no reason to be close to $\frac{S_n}{\sqrt{n}}$.

6.1.1 Preparation

We work with the model $(\Omega, \mathcal{F}, \mathbb{P})$.

Our derivation is based on the use of the counting processes $N_n(i)$. Recall that $N_0(i) = 0$ and that $N_n(i) := \sum_{1 \leq m \leq n} \epsilon_m(i)$ for $n \geq 1$, where $\epsilon_n(i) := \mathbb{1}_{I_n=i}$ is 1 if the n^{th} partial measurement outcome is i and 0 otherwise.

We start by listing some properties of these counting processes and their relationship to the solution of (3). Then we shall formulate and prove the analogous statements for the continuous time limit.

It is obvious that the filtration $\mathcal{F}_0, \mathcal{F}_1, \dots$ is the natural filtration of the vector counting processes N_n .

Also recall that the random recursion relation (3) can be solved in terms of the counting processes as

$$Q_n(\alpha) = q_0(\alpha) \frac{\prod_i p(i|\alpha)^{N_n(i)}}{\sum_{\beta} q_0(\beta) \prod_i p(i|\beta)^{N_n(i)}}.$$

A trivial but crucial observation is that under each \mathbb{P}_α , N_n is the sum of independent identically distributed (i.i.d) random vectors.

As a first consequence, a simple computation leads to

$$\begin{aligned} \mathbb{E} \left(e^{\sum_{l=1}^k \sum_i \lambda_l(i)(N_{n_l}(i) - N_{n_{l-1}}(i))} \right) &= \\ \sum_{\alpha} q_0(\alpha) \prod_{l=1}^k \left(\sum_i e^{\lambda_l(i)} p(i|\alpha) \right)^{n_l - n_{l-1}} \end{aligned} \quad (18)$$

for $k \geq 1$, arbitrary non-decreasing sequences of integers $0 = n_0 \leq n_1 \leq \dots \leq n_k$ of length k , and arbitrary (complex) $\lambda_l(i)$'s. A second consequence is that under \mathbb{P}_α each $N_n(i)$ is a sub-martingale and $N_n(i) = (N_n(i) - np(i|\alpha)) + np(i|\alpha)$ is its Doob decomposition as a martingale plus a predictable (in that case deterministic) increasing process. Moreover, if $n \geq 1$, and if X is an \mathcal{F}_{n-1} measurable random variable, we compute

$$\begin{aligned} \mathbb{E}(X \epsilon_n(i)) &= \sum_{\alpha} q_0(\alpha) \mathbb{E}_{\alpha}(X \epsilon_n(i)) \\ &= \sum_{\alpha} q_0(\alpha) \mathbb{E}_{\alpha}(X) p(i|\alpha) = \sum_{\alpha} \mathbb{E}(X Q_{n-1}(\alpha)) p(i|\alpha). \end{aligned}$$

For the last equality we used the Q 's characterization as Radon-Nikodym derivatives. This proves that $\mathbb{E}(\epsilon_n(i) | \mathcal{F}_{n-1}) = \sum_{\alpha} Q_{n-1}(\alpha) p(i|\alpha) = \pi_n(i)$. Hence, setting

$$A_n(i) := \sum_{m=1}^n \pi_m(i),$$

an increasing predictable process, we find that $X_n(i) := N_n(i) - A_n(i)$ is an \mathcal{F}_n -martingale under \mathbb{P} , so each $N_n(i)$ is again a sub-martingale with Doob

decomposition

$$N_n(i) = X_n(i) + A_n(i) \quad (19)$$

under \mathbb{P} .

Finally, by some simple algebra we may rephrase the random recursion relation satisfied by the Q 's as a stochastic difference equation

$$Q_n(\alpha) - Q_{n-1}(\alpha) = Q_{n-1}(\alpha) \sum_i \frac{p(i|\alpha)}{\pi_{n-1}(i)} (X_n(i) - X_{n-1}(i)). \quad (20)$$

6.1.2 Derivation of the pointer state distribution evolution

Equation (20) admits eq.(15) as a naive continuous time limit when δ , the scaling parameter, goes to 0^+ . To put the validity of this formal approach on a firmer ground, one needs to prove the existence of a continuous time limit. This is a classical topic, but the presence of the scaling parameter δ in various places prevents us from applying standard theorems straightforwardly. So we rely on a down-to-earth approach, which is rather technical. For this reason we relegated the argument to appendix B. This is where the interested reader should look for some background, precise definitions, etc. We give here a brief summary:

- By an appropriate interpolation procedure, one defines a δ -dependent push-forward $\mu_\alpha(\delta)$ of each \mathbb{P}_α and $\mu(\delta)$ of \mathbb{P} in $C_0(\mathbb{R}^+, \mathbb{R}^I)$, the space of continuous functions from \mathbb{R}^+ to \mathbb{R}^I vanishing at 0.

- We are not able to prove the convergence in law of the $\mu_\alpha(\delta)$ or of $\mu(\delta)$ when $\delta \rightarrow 0^+$.

- However, the finite dimensional distributions of the joint processes $N_n(i)$ and $Q_n(\alpha)$ under each \mathbb{P}_\bullet (where \bullet stands either for an element of \mathcal{S} or for nothing) admit, after appropriate time dependent centering and scaling, continuous time limits which are the joint finite dimensional distributions, under a probability measure μ_\bullet on $C_0(\mathbb{R}^+, \mathbb{R}^I)$, for processes $W_t(i)$, to be thought of as

$$\lim_{\delta \rightarrow 0^+} \sqrt{\delta} (N_{t/\delta}(i) - p_0(i)t/\delta),$$

and $Q_t(\alpha)$, to be thought of as $\lim_{\delta \rightarrow 0^+} Q_{t/\delta}(\alpha)$.

- The process W_t is the canonical coordinate process on $C(\mathbb{R}^+, \mathbb{R}^I)$, and its natural filtration \mathcal{G}_t is to be thought of as the continuous time limit of the natural filtration for N_n , i.e. as the information collected by indirect measurements up to time t .

- The identity $\mu = \sum_\alpha q_0(\alpha) \mu_\alpha$ holds. The Radon-Nikodym derivative of $\mu(\alpha)$ with respect to μ on \mathcal{G}_t is $M_t(\alpha)/M_t$ where

$$M_t(\alpha) := e^{\sum_i \Gamma(i|\alpha) W_t(i) - \frac{t}{2} \sum_i p_0(i) \Gamma(i|\alpha)^2}, \quad M_t := \sum_\alpha q_0(\alpha) M_t(\alpha)$$

For each α , $M_t^{-1}(\alpha)$ is a \mathcal{G}_t -martingale under μ_α , and M_t^{-1} is a \mathcal{G}_t -martingale under μ .

– For each $T > 0$, under the measure $M_T^{-1}d\mu$ (which coincides with $M_T^{-1}(\alpha)d\mu_\alpha$ for every α), the process $(W_t)_{t \in [0, T]}$ is a continuous time-homogeneous centered Gaussian process with covariance $\min(t, s)(\delta_{i,j} p_0(i) - p_0(i)p_0(j))$. Thus, by Girsanov's theorem, under each μ_α , W_t is a continuous time-homogeneous Gaussian process with independent increments, characterized by

$$\begin{aligned}\mathbb{E}^{\mu_\alpha}(W_t(i)) &= tp_0(i)\Gamma(i|\alpha) \\ \text{Cov}^{\mu_\alpha}(W_t(i), W_s(j)) &= \min(t, s)(\delta_{i,j} p_0(i) - p_0(i)p_0(j)).\end{aligned}$$

– There is an explicit formula for the Q_t 's in terms of the W_t 's, namely:

$$Q_t(\alpha) = q_0(\alpha) \frac{M_t(\alpha)}{M_t} = q_0(\alpha) \frac{e^{\sum_i \Gamma(i|\alpha)W_t(i) - \frac{t}{2} \sum_i p_0(i)\Gamma(i|\alpha)^2}}{\sum_\beta q_0(\beta) e^{\sum_i \Gamma(i|\beta)W_t(i) - \frac{t}{2} \sum_i p_0(i)\Gamma(i|\beta)^2}}.$$

We are now in position to check that all the properties established in the discrete setting, as listed in section 6.1.1, have a direct naive counterpart in the continuous time setting.

The construction of the filtration \mathcal{G}_t as the natural filtration for the canonical process was already explained. We have also already mentioned that there is an explicit formula for the Q_t 's. The counterpart of (18), the Laplace transform of the counting processes joint distributions is given for the canonical process in eq.(29), Appendix B.

The counterpart of the counting process Doob decomposition under \mathbb{P}_α is $W_t(i) = (W_t(i) - tp_0(i)\Gamma(i|\alpha)) + tp_0(i)\Gamma(i|\alpha)$ under μ_α .

To get the counterpart of the counting process Doob-Meyer decomposition under \mathbb{P} , i.e. the Doob-Meyer decomposition of $W_t(i)$ under μ , we use Girsanov's theorem. As recalled above, for every $T > 0$ $(W_t(i))_{t \in [0, T]}$ is a continuous martingale under $M_T^{-1}d\mu$. From

$$dM_t/M_t = \sum_\alpha q_0(\alpha) \frac{M_t(\alpha)}{M_t} \sum_i \Gamma(i|\alpha) dW_t(i) = \sum_\alpha Q_t(\alpha) \sum_i \Gamma(i|\alpha) dW_t(i),$$

we infer that the increasing process

$$A_t(i) := \int_0^t ds \sum_\alpha Q_s(\alpha) p_0(i) \Gamma(i|\alpha)$$

is the compensator of $W_t(i)$, i.e.

$$X_t(i) := W_t(i) - A_t(i)$$

is a \mathcal{G}_t martingale under μ , with quadratic variation given by (16). It is easily seen that $A_t(i), X_t(i)$ are the obvious continuous time limits of $A_n(i), X_n(i)$.

It remains to write down the stochastic evolution equations for the Q_t 's. By Itô's formula for a ratio, we find

$$\frac{dQ_t(\alpha)}{Q_t(\alpha)} = \left(\frac{dM_t(\alpha)}{M_t(\alpha)} - \frac{dM_t}{M_t} \right) \left(1 - \frac{dM_t}{M_t} \right).$$

leading immediately to (15) which we reproduce for convenience:

$$dQ_t(\alpha) = Q_t(\alpha) \sum_i (\Gamma(i|\alpha) - \langle \Gamma_i \rangle_t) dX_t(i)$$

where $\langle \Gamma_i \rangle_t := \sum_\beta Q_t(\beta) \Gamma(i|\beta)$. Note again that this equation is also the naive continuous time limit of the discrete equation (20).

To summarize, one makes no mistakes if one works naively and forgets about the lengthy rigorous construction of the continuous time limit. This gives us confidence in what follows to proceed straightforwardly in the derivation of continuous time equations in more complicated situations.

6.1.3 Convergence of the continuous time evolution

We now prove the convergence of $Q_t(\alpha)$ when t goes to infinity. Its almost sure convergence is a direct consequence of its martingale property. We need to prove that the final distribution is

$$Q_\infty(\alpha) = \begin{cases} 0 & \text{if } \alpha \notin \underline{\Upsilon} \\ q_0(\alpha)/\bar{q}_0(\underline{\alpha}) & \text{if } \alpha \in \underline{\Upsilon} \end{cases} \quad (21)$$

and that the convergence is exponential with the characteristic time $\tau_{\Upsilon\alpha}$.

First we prove that the limit of the sector distribution $\bar{Q}_t(\underline{\alpha}) := \sum_{\alpha \in \underline{\alpha}} Q_t(\alpha)$ is

$$\bar{Q}_\infty(\underline{\alpha}) = \delta_{\underline{\alpha}, \underline{\Upsilon}}$$

The equation of evolution for the sector distribution is

$$d\bar{Q}_t(\underline{\alpha}) = \sum_{\alpha' \in \underline{\alpha}} dQ_t(\alpha') = \bar{Q}_t(\underline{\alpha}) \sum_i (\Gamma(i|\alpha) - \langle \Gamma_i \rangle_t) dX_t(i)$$

In the limit $t \rightarrow \infty$, we have $Q_\infty(\underline{\alpha})(\Gamma(i|\alpha) - \langle \Gamma_i \rangle_\infty) = 0$ for all i , μ -almost surely. Then either $\bar{Q}_\infty(\underline{\alpha}) = 0$ or $\Gamma(i|\alpha) = \sum_i \bar{Q}_\infty(\underline{\beta}) \Gamma(i|\beta)$. Since $\Gamma(i|\alpha) \neq \Gamma(i|\beta)$ if $\underline{\alpha} \neq \underline{\beta}$, the solution to the limit equation is $\bar{Q}_\infty(\underline{\alpha}) = \delta_{\underline{\alpha}, \underline{\Upsilon}}$.

Second we show that $\frac{Q_t(\alpha')}{Q_t(\alpha)} = \frac{q_0(\alpha')}{q_0(\alpha)}$ if α', α are in the same sector. As in the discrete case, this relation implies eq.(21). We compute:

$$\frac{Q_t(\alpha)}{Q_t(\alpha')} d\frac{Q_t(\alpha')}{Q_t(\alpha)} = \frac{dQ_t(\alpha')}{Q_t(\alpha')} - \frac{dQ_t(\alpha)}{Q_t(\alpha)} + \left(\frac{dQ_t(\alpha)}{Q_t(\alpha)} \right)^2 - \frac{dQ_t(\alpha')}{Q_t(\alpha')} \frac{dQ_t(\alpha)}{Q_t(\alpha)}$$

Since, $\frac{dQ_t(\alpha')}{Q_t(\alpha')} = \frac{dQ_t(\alpha)}{Q_t(\alpha)}$ for α and α' in the same sector, we obtain $d\frac{Q_t(\alpha')}{Q_t(\alpha)} = 0$ if $\alpha, \alpha' \in \underline{\alpha}$. For all time t ,

$$\frac{Q_t(\alpha')}{Q_t(\alpha)} = \frac{q_0(\alpha')}{q_0(\alpha)} \text{ if } \alpha \sim \alpha'$$

which achieves the proof for the limit pointer state distribution.

Finally we prove the exponential convergence. The tools we use are the convergence of the pointer state distribution and the Itô calculus. From the distribution convergence, we have $\langle \Gamma_i \rangle_t \simeq \Gamma(i|\Upsilon)$ for t large enough. The evolution equation (15) for $\alpha \notin \underline{\Upsilon}$ becomes $dQ_t(\alpha) \simeq Q_t(\alpha) \sum_i (\Gamma(i|\alpha) - \Gamma(i|\Upsilon)) dX_t(i)$. This equation is a well known stochastic exponential equation. Thus, at large time t , with good approximation

$$Q_t(\alpha) \simeq \text{const.} e^{-\frac{t}{2} \sum_i p_0(i) (\Gamma(i|\alpha) - \Gamma(i|\Upsilon))^2 + \sum_i X_t(i) (\Gamma(i|\alpha) - \Gamma(i|\Upsilon))}$$

for $\alpha \notin \underline{\Upsilon}$. Keeping only the leading term in the exponential we obtain the exponential decrease, $Q_t(\alpha) \simeq \exp(-t/\tau_{\Upsilon\alpha})$, with $\tau_{\Upsilon\alpha}$ given in eq.(17).

6.1.4 Different partial measurement methods

The previous results can easily be extended to cases where different measurement methods are randomly used. Since proofs are similar to those of previous sections, here we only present a general outline of the approach. We limit ourselves to a time and realization independent partial measurement method distribution. In this case $d_n(o_1, i_1, \dots, i_n, o_{n+1}) = \prod_{k=1}^{n+1} c(o_k)$ with $\sum_o c(o) = 1$.

To stay in the scope of the diffusive limit we assume that for any o , $\langle i|\Psi^{(o)} \rangle \neq 0$.

Following previous sections, we define linear interpolations $W_t^{(\delta)}(o, i)$ of the counting processes which naively read

$$\sqrt{\delta}(N_{t/\delta}(o, i) - c(o)p_0^o(i)t/\delta)$$

See Appendix C for precise definitions. As shown in this appendix, all finite dimensional distribution functions of $W_t^{(\delta)}(o, i)$ under (a push-forward of) \mathbb{P}_α (resp. \mathbb{P}) have a finite limit as $\delta \rightarrow 0^+$ which coincide with those of continuous random processes, denoted $W_t(o, i)$, under appropriate measures denoted μ_α (resp. μ). Under μ_α , $W_t(o, i)$ is a Gaussian process with

$$\begin{aligned} \mathbb{E}^{\mu_\alpha}(W_t(o, i)) &= t c(o) p_0^o(i) \Gamma^{(o)}(i|\alpha) \\ \text{Cov}^{\mu_\alpha}(W_t(o, i), W_s(o', j)) &= \min(t, s) (c(o)p_0^o(i)\delta_{(o,i),(o',j)} - c(o)p_0^o(i)c(o')p_0^{o'}(j)). \end{aligned}$$

with $p_0^o(i) = |\langle i|\Psi^{(o)} \rangle|^2$ and $\Gamma^{(o)}(i|\alpha) = 2\text{Im} \left(\frac{\langle i|H_\alpha^{(o)}|\Psi^{(o)} \rangle}{\langle i|\Psi^{(o)} \rangle} \right)$.

The measure μ is the sum $\mu = \sum_{\alpha} q_0(\alpha) \mu_{\alpha}$. The Radon-Nikodym derivative of $\mu(\alpha)$ with respect to μ is $M_t(\alpha)/M_t$ where $M_t = \sum_{\alpha} q_0(\alpha) M_t(\alpha)$ with

$$M_t(\alpha) = e^{\sum_{(o,i) \in E} \Gamma^{(o)}(i|\alpha) W_t(o,i) - \frac{t}{2} \sum_{(o,i) \in E} c(o) p_0^o(i) \Gamma^{(o)}(i|\alpha)^2}$$

As in the section 6.1.2 we define

$$X_t(o, i) = W_t(o, i) - \int_0^t \sum_{\alpha} Q_s(\alpha) c(o) p_0^o(i) \Gamma^{(o)}(i|\alpha) ds$$

The $X_t(o, i)$ are martingales under μ . From this definition we obtain straightforwardly

$$dQ_t(\alpha) = Q_t(\alpha) \sum_{(o,i) \in E} (\Gamma^{(o)}(i|\alpha) - \langle \Gamma^{(o)}(i) \rangle_t) dX_t(o, i)$$

with $\langle \Gamma^{(o)}(i) \rangle_t = \sum_{\alpha} \Gamma^{(o)}(i|\alpha) Q_t(\alpha)$ and

$$dX_t(o, i) dX_t(o', j) = dt (c(o) p_0^o(i) \delta_{(o,i), (o',j)} - c(o) p_0^o(i) c(o') p_0^{o'}(j))$$

The limit of $Q_t(\alpha)$ is the same but the sectors are now the sets of basis states such that $\Gamma^{(o)}(i|\alpha') = \Gamma^{(o)}(i|\alpha)$ for all partial measurement methods and all partial measurement results. The convergence toward the limit distribution is exponential

$$Q_t(\alpha) \simeq \exp \left[-\frac{t}{2} \sum_{(o,i) \in E} c(o) p_0^o(i) (\Gamma^{(o)}(i|\alpha) - \Gamma^{(o)}(i|\Upsilon))^2 \right]$$

The approximation hold if t is large enough. The convergence is exponential with a characteristic time

$$\frac{2}{\tau_{\Upsilon\alpha}} = \sum_{(o,i) \in E} c(o) p_0^o(i) (\Gamma^{(o)}(i|\alpha) - \Gamma^{(o)}(i|\Upsilon))^2$$

We find a convergence rate which is a mean convergence rate as in the discrete case. The same result is found by taking the scaling limit of the discrete case mean relative entropy.

6.2 Density matrix evolution

We are now interested in the density matrix evolution.

As in section 5.2, the density matrix at time n can be decomposed in the basis of pointer states:

$$\rho_n = \sum_{\alpha, \beta} A_n(\alpha, \beta) |\alpha\rangle \langle \beta|$$

The same decomposition applies to the time continuous density matrix we will define. The recurrence relation (1) translates for $A_n(\alpha, \beta)$ in

$$A_n(\alpha, \beta) = \frac{A_{n-1}(\alpha, \beta) M^{(o_n)}(i_n|\alpha) M^{(o_n)}(i_n|\beta)^*}{\sum_{\gamma} q_{n-1}(\gamma) p^{o_n}(i_n|\gamma)}$$

Where $M^{(o)}(i|\alpha) = \langle i|U^{(o)}(\alpha)|\Psi^{(o)}\rangle$. For $\alpha = \beta$, this reproduces the pointer state distribution recurrence relation (3), as expected.

We first limit ourselves to the case where only one partial measurement method is used and we omit the index o . The results will then be generalized to different partial measurement methods. We used a few hypotheses to get the continuous-time limit:

- The first two assumptions are related to the development in $\sqrt{\delta}$ of the conditional probabilities $p(i|\alpha)$. As stated before, the interaction Hamiltonian must be rescaled $H_I \rightarrow \frac{1}{\sqrt{\delta}} H_I$ and for any partial measurement result i , $\langle i|\Psi\rangle \neq 0$. Then

$$p(i|\alpha) = p_0(i)(1 + \sqrt{\delta} \Gamma_{\delta}(i|\alpha))$$

with $p_0(i) = |\langle i|\psi\rangle|^2$. The assumption $\langle i|\Psi\rangle \neq 0$ leads to the diffusive limit. If this condition is not fulfilled for every i , then a jump-diffusion limit is found as shown in [23].

- A third assumption is needed to obtain a convergence of the evolution of the phases between different pointer states. The interaction Hamiltonian expectation must be zero :

$$\langle \Psi|H_I|\Psi\rangle = 0$$

Under these assumptions, we show in Appendix D that the time continuous evolution derived from the discrete time case is

$$A_t(\alpha, \beta) = A_0(\alpha, \beta) \frac{e^{l(\alpha, \beta)t - i \sum_i (c(i|\alpha) - c(i|\beta)^*) W_t(i)}}{\sum_{\gamma} q_0(\gamma) e^{\sum_i -i \Gamma(i|\gamma) W_t(i) - \frac{t}{2} p_0(i) \Gamma(i|\gamma)^2}} \quad (22)$$

with $c(i|\alpha) = \frac{\langle i|H_I|\Psi\rangle}{\langle i|\Psi\rangle}$ and

$$l(\alpha, \beta) := -i(E_{\alpha} - E_{\beta}) - \frac{1}{2} \sum_i p_0(i) (|c(i|\alpha)|^2 + |c(i|\beta)|^2 - c(i|\alpha)^2 - c(i|\beta)^2)$$

If we set $\alpha = \beta$ we recover the result on the pointer state distribution.

A simple computation using Itô rules shows that this process is solution of a Belavkin diffusive equation:

$$d\rho_t = L(\rho_t) - i \sum_i (C_i \rho_t - \rho_t C_i^{\dagger} - \rho_t \text{Tr}[(C_i - C_i^{\dagger})\rho_t]) dX_t(i) \quad (23)$$

with the Lindbladian

$$L(\rho) = -i[H_s, \rho] + \sum_i p_0(i)(C_i \rho C_i^\dagger - \frac{1}{2}\{C_i^\dagger C_i, \rho\})$$

and $C_i := \sum_\alpha c(i|\alpha)|\alpha\rangle\langle\alpha| = \frac{\langle i|H_I|\Psi\rangle}{\langle i|\Psi\rangle}$.

As shown in [20], this equation corresponds to the time continuous limit of repeated POVM processes (1) even if the non destruction assumption (2) is not fulfilled.

In the next section we study the long time behavior of such evolution in the non destructive case.

6.2.1 Long time convergence of the density matrix

The pointer state distribution convergence indicates that, in the long time limit, the system is in a subspace of basis $\underline{\Upsilon}$. This information only tells us what is the limit of the elements $A_t(\alpha, \beta)$ when α or β are not in the limit sector $\underline{\Upsilon}$. From the Cauchy-Schwarz theorem, $\lim_{t \rightarrow \infty} Q_t(\alpha)Q_t(\beta) = 0$ implies $\lim_{t \rightarrow \infty} A_t(\alpha, \beta) = 0$. For the elements $A_t(\alpha, \beta)$ with $\alpha, \beta \in \underline{\Upsilon}$, the limit $t \rightarrow \infty$ is yet unknown.

We decompose the operators C_i in a sum of two hermitian operators

$$C_i = R_i + iS_i$$

with $R_i = \sum_\alpha \text{Re}(c(i|\alpha))|\alpha\rangle\langle\alpha|$ and $S_i = \sum_\alpha \frac{1}{2}\Gamma(i|\alpha)|\alpha\rangle\langle\alpha|$.

As in the discrete time case, the density matrix evolution has to be modified by a unitary process in order to get convergence when t goes to infinity. Let \tilde{U}_t be the unitary diagonal operator defined via

$$\tilde{U}_t^{-1} d\tilde{U}_t = -i(H_s - \sum_i p_0(i)[R_i(S_i - 2\langle S_i \rangle_t) - \frac{i}{2}R_i^2])dt - i \sum_i R_i dX_t(i)$$

and let $\tilde{\rho}_t$ be the modified density matrix

$$\tilde{\rho}_t = \tilde{U}_t^\dagger \rho_t \tilde{U}_t$$

As we show below it has an almost sure limit

$$\lim_{t \rightarrow \infty} \tilde{\rho}_t = \frac{1}{q_0(\underline{\Upsilon})} P_{\underline{\Upsilon}} \rho_0 P_{\underline{\Upsilon}} \quad (24)$$

where $P_{\underline{\Upsilon}} := \sum_{\gamma \in \underline{\Upsilon}} |\gamma\rangle\langle\gamma|$ is the projector on the subspace corresponding to the sector $\underline{\Upsilon}$. Therefore, $\tilde{\rho}_\infty$ is equivalent to the density matrix we would have found if an initial Von Neumann measurement had been performed on the system. The unitary operator \tilde{U}_t only induces a rotation inside the limit subspace.

Recall that we only need to prove the convergence of the $\tilde{\rho}_t$ matrix elements corresponding to two pointer states in the same sector. From the Belavkin equation (23) and using Itô rules, we find the evolution equation for $\tilde{\rho}_t$:

$$d\tilde{\rho}_t = \sum_i p_0(i) (S_i \tilde{\rho}_t S_i - \frac{1}{2} \{S_i S_i, \tilde{\rho}_t\}) dt - i \sum_i (\{S_i, \tilde{\rho}_t\} - 2\text{Tr}[S_i \tilde{\rho}_t]) dX_t(i)$$

Thus, the time evolution of matrix elements $\tilde{A}_t(\alpha, \beta)$ of $\tilde{\rho}_t$ with β and α in the same sector is,

$$d\tilde{A}_t(\alpha, \beta) = \tilde{A}_t(\alpha, \beta) \sum_i (\Gamma(i|\alpha) - \langle \Gamma(i) \rangle_t) dX_t(i)$$

Noticing that $Q_t(\alpha) d\tilde{A}_t(\alpha, \beta) = \tilde{A}_t(\alpha, \beta) dQ_t(\alpha)$ and repeating the discussion of section 6.1.3, we get

$$\tilde{A}_\infty(\alpha, \beta) = \begin{cases} \frac{A_0(\alpha, \beta)}{q_0(\underline{\Upsilon})} & \text{if } \alpha \text{ and } \beta \in \underline{\Upsilon} \\ 0 & \text{else} \end{cases}$$

This proves the limit (24).

6.2.2 Extension to different partial measurement methods

We can extend our results to cases where different partial measurement methods are used. Once again we limit ourselves to time independent random protocols. The density matrix evolution is modified as follows:

$$d\rho_t = L(\rho_t) dt + \sum_{(o,i) \in E} \mathcal{D}_{(o,i)}(\rho_t) dX_t(o, i)$$

with

$$\mathcal{D}_{(o,i)}(\rho_t) = -i(C_i^{(o)} \rho_t - \rho_t C_i^{(o)\dagger} - \rho_t \text{Tr}[C_i^{(o)} \rho_t - \rho_t C_i^{(o)\dagger}])$$

where $C_i^{(o)} = \frac{\langle i|H_f^{(o)}|\Psi_o\rangle}{\langle i|\Psi_o\rangle}$ and

$$L(\rho_t) = -i[H_s, \rho_t] + \sum_{(o,i) \in E} c(o) p_0^o(i) (C_i^{(o)} \rho_t C_i^{(o)\dagger} - \frac{1}{2} \{C_i^{(o)\dagger} C_i^{(o)}, \rho_t\})$$

As before $c(o)$ is the probability of using measurement method o . The limit density matrix can be analyzed as above: we obtain identical convergence statements once the density matrix has been rotated using an appropriate unitary \tilde{U}_t .

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A Details for mutual singularity

We prove that if o is recurrent but (o, i) is not then $\sum_{\gamma} Q(\gamma) p^o(i|\gamma) = 0$.

Observe that under \mathbb{P}_{α} we have the Markov property

$$\mathbb{E}_{\alpha}(\mathbb{1}_{I_n=i} | \mathcal{F}_{n-1}) = p^{O_n}(i|\alpha) \quad (25)$$

It says that, under \mathbb{P}_{α} , $O_n = o$ the next measurement outcome is $i \in \text{spec}(o)$ with probability $p^o(i|\alpha)$ independently of what happened before.

Now assume that under \mathbb{P}_{α} measurement method o is recurrent with probability 1. Take $0 \leq T_1 < T_2 < \dots < T_k \dots$ to be the times when the measurement method is o . We show, using the strong Markov property, that I_{T_1}, I_{T_2}, \dots are independent identically distributed random variables with distribution $p^o(\cdot|\alpha)$. This is quite natural: the functions d_n help choosing the measurement method, but they do not influence the measurement result.

Indeed, note first that the above statement is trivial when there is only one measurement method, because then there is no need to invoke stopping times and the strong Markov property. In the general case, note the slight mismatch with usual notations: $\{T_k \leq n\}$ is in fact \mathcal{F}_{n-1} measurable, so it is natural to write \mathcal{F}_{T_k-1} for the algebra associated to the stopping time T_k . Then write

$$\begin{aligned} \mathbb{E}_{\alpha}(\mathbb{1}_{I_{T_1}=i_1} \dots \mathbb{1}_{I_{T_k}=i_k} | \mathcal{F}_{T_k-1}) &= \mathbb{1}_{I_{T_1}=i_1} \dots \mathbb{1}_{I_{T_{k-1}}=i_{k-1}} \mathbb{E}_{\alpha}(\mathbb{1}_{I_{T_k}=i_k} | \mathcal{F}_{T_k-1}) \\ &= \mathbb{1}_{I_{T_1}=i_1} \dots \mathbb{1}_{I_{T_{k-1}}=i_{k-1}} p^o(i_k|\alpha) \end{aligned}$$

One can go on to condition with respect to $\mathcal{F}_{T_{k-1}-1}, \dots$ until one finds the plain expectation

$$\mathbb{E}_{\alpha}(\mathbb{1}_{I_{T_1}=i_1} \dots \mathbb{1}_{I_{T_k}=i_k}) = p^o(i_1|\alpha) \dots p^o(i_k|\alpha).$$

As a consequence, for any α such that o is recurrent under \mathbb{P}_{α} :

– either $p^o(i|\alpha) > 0$ and with \mathbb{P}_{α} -probability 1 the outcome i appears infinitely many times in the sequence I_{T_1}, I_{T_2}, \dots , i.e. (o, i) is recurrent with probability 1,

– or $p^o(i|\alpha) = 0$ and i never appears in the sequence I_{T_1}, I_{T_2}, \dots , i.e. (o, i) never appears.

Now assume that o is recurrent under all \mathbb{P}_{α} 's. The above implies immediately that the probability under \mathbb{P} that (o, i) is non-recurrent (this event is denoted by $\tilde{A}_{(o,i)}$) is given by $\sum_{\gamma, p^o(i|\gamma)=0} q_0(\gamma)$.

If $p^o(i|\beta) = 0$ then $Q(\beta) = Q(\beta) \mathbb{1}_{\tilde{A}_{(o,i)}}$ because by the recursion relation $Q_n(\beta) = 0$ whenever (o, i) has shown up before time n . So

$$\mathbb{E}(Q(\beta) | \tilde{A}_{(o,i)}) = \frac{\mathbb{E}(Q(\beta))}{\mathbb{E}(\mathbb{1}_{\tilde{A}_{(o,i)}})} = \frac{q_0(\beta)}{\sum_{\gamma, p^o(i|\gamma)=0} q_0(\gamma)}$$

which implies that

$$\mathbb{E}\left(\sum_{\gamma, p^o(i|\gamma)=0} Q(\gamma) | \tilde{A}_{(o,i)}\right) = 1$$

Hence, conditional on $\tilde{A}_{(o,i)}$, the $Q(\alpha)$'s for which $p^o(i|\alpha) > 0$ have to vanish. Equivalently, $Q(\gamma)p^o(i|\gamma) = 0$ for each γ and $\sum_{\gamma} Q(\gamma)p^o(i|\gamma) = 0$, which was to be proved.

B Proof of existence of a continuous time limit

We first put the notion of continuous time limit in context.

Let V be the vector space $C_0(\mathbb{R}^+, \mathbb{R}^I)$ of continuous functions f from \mathbb{R}^+ to \mathbb{R}^I such that $f_0 = 0 \in \mathbb{R}^I$. For each $\delta > 0$, and each $\omega \in \Omega$ we define a function $W_t^{(\delta)}(i)$ on \mathbb{R}^+ by linear interpolation of $W_t^{(\delta)}(i) := \sqrt{\delta}(N_{t/\delta}(i) - p_0(i)t/\delta)$ if t/δ is an integer. Explicitly, for $t \in [\delta n, \delta(n+1)]$

$$W_t^{(\delta)}(i) = \sqrt{\delta}((n+1 - t/\delta)N_n(i) + (t/\delta - n)N_{n+1}(i) - p_0(i)t/\delta).$$

For every $\omega \in \Omega$ the function $W_t^{(\delta)}(i)$ is continuous for $t \in \mathbb{R}^+$. So we have a map $W^{(\delta)} : \Omega \rightarrow V$. But, as already pointed out before, there is no hope that, for a fixed $\omega \in \Omega$, $W_t^{(\delta)}(i)$ has a limit when $\delta \rightarrow 0^+$. The only clear fact is that for a fixed t , the central limit theorem ensures that the distribution of $W_t^{(\delta)}(i)$ under each \mathbb{P}_α has a Gaussian limit when $\delta \rightarrow 0^+$. Note that this observation fixes the scaling $\sqrt{\delta}$ as the only one possible.

But if we are interested in convergence as a process, a deeper approach is needed. If we endow V with the topology of uniform convergence on compact sets $\mathcal{T}(V)$ and with the corresponding Borel σ -algebra $\mathcal{B}(V)$, we can show that the map $W^{(\delta)}$ is measurable from (Ω, \mathcal{F}) to $(V, \mathcal{B}(V))$. This is not difficult, because by a classical result, $\mathcal{B}(V)$ is the smallest σ -algebra on V containing the family of sets

$$B^{t,i,a} := \{f \in C_0(\mathbb{R}^+, \mathbb{R}^I), a < f_t(i)\}$$

indexed by $t \in]0, +\infty[$, $i \in I$ and $a \in \mathbb{R}$. It is plain that the inverse image of $B^{t,i,a}$ under $W^{(\delta)}$ is in \mathcal{F}_n whenever $n > t/\delta$. As the filtration on Ω is exactly the one making the N_n \mathcal{F}_n -measurable, the appropriate filtration on $C(\mathbb{R}^+, \mathbb{R}^I)$ should be the natural one, the smallest making the canonical process adapted⁶. We denote it by \mathcal{G}_t .

Then any probability measure on (Ω, \mathcal{F}) induces via $W^{(\delta)}$ a probability measure on $(V, \mathcal{B}(V))$. Note that, via (14), the measures we defined previously on (Ω) depend on δ , and to be explicit we write $\mathbb{P}^{(\delta)}$, $\mathbb{E}^{(\delta)}$, etc. to stress this fact. Let $\mu_\alpha(\delta)$ be the image measure of $\mathbb{P}_\alpha^{(\delta)}$ pushed forward by $W^{(\delta)}$ on

⁶Of course, as long as $\delta > 0$, W_t looks a bit forward in the future, as it involves N_{n+1} for $t \in [\delta n, \delta(n+1)]$, but for continuous process this does not matter.

$(V, \mathcal{B}(V))$. As $(V, \mathcal{T}(V))$ is a so-called Polish space, there is a nice notion of convergence for measures on it, called “weak convergence” of measures, and we could ask if the $\mu_\alpha(\delta)$ ’s converge weakly to some probability measure on $(V, \mathcal{B}(V))$ (and then, so would $\mu(\delta) = \sum_\alpha q_0(\alpha) \mu_\alpha(\delta)$). Note that despite its name, weak convergence is strong enough to ensure the convergence of the expectations of rather general functionals, so we could hope to control the Q ’s continuous time limit as well, because they are nice functionals of the counting process.

It is usually in this context that continuous time limits have a meaning. In this setting, there are a number of theorems, called functional central limit theorems, or Donsker invariance principles, that express the continuous time limit of random walks (with independent increments) in terms of Brownian motions in fine details. Alas, though under each \mathbb{P}_α , $N_n(i)$ is a random walk with independent increments, the theorems we are aware do not apply immediately. The problem is that the δ dependence is not only in W , but also in the \mathbb{P}_α ’s. This problems would show up even more dramatically to deal with the Q ’s convergence and there relationships with W . While we think these are purely technical details in our case, we shall not try to deal with them: we shall instead rely on a weaker and slightly less natural notion of continuous time limit that will suffice for our purposes. To say things in more mathematical terms: we shall content ourselves with proving that the joint finite dimensional distributions of W ’s and Q ’s converge to joint finite dimensional distributions of continuous processes we can identify explicitly, but we do not embark on the more technical task of proving tightness.

We now turn to our explicit approach of the continuous time limit.

From the characteristic function (18) we obtain easily that

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \mathbb{E}^{(\delta)} \left(e^{\sum_{l=1}^k \sum_i \lambda_l(i) (W_{t_l}^{(\delta)}(i) - W_{t_{l-1}}^{(\delta)}(i))} \right) = \\ \left(\sum_\alpha q_0(\alpha) e^{\sum_{l=1}^k (t_l - t_{l-1}) \sum_i \lambda_l(i) p_0(i) \Gamma(i|\alpha)} \right) \times \\ e^{\frac{1}{2} \sum_{l=1}^k (t_l - t_{l-1}) (\sum_i p_0(i) \lambda_l(i)^2 - (\sum_i p_0(i) \lambda_l(i))^2)} \quad (26) \end{aligned}$$

for $k \geq 1$, arbitrary non-decreasing sequences $0 = t_0 \leq t_1 \leq \dots \leq t_k$ of length k and arbitrary (complex) $\lambda_l(i)$ ’s.

By a standard theorem on characteristic functions, we have thus proved that the $W_t^{(\delta)}(i)$ finite marginals (under each $\mathbb{P}_\alpha^{(\delta)}$ and under $\mathbb{P}^{(\delta)}$) have a limit for $\delta \rightarrow 0^+$. This is much weaker than what weak convergence of measures would ensure. It is enough to ensure that the limit marginals satisfy the Kolmogorov consistency criterion, but it does not guarantee that it is possible to concentrate the corresponding process on $C_0(\mathbb{R}^+, \mathbb{R}^I)$. However in the case at hand, we can bypass this problem because of the simple form of the result, which is Gaussian for each α .

Let ν be the Wiener measure of a standard Brownian motion on $C_0(\mathbb{R}^+, \mathbb{R}^I)$. The linear map from \mathbb{R}^I to itself defined by

$$y(i) := \sqrt{p_0(i)}(x(i) - \sqrt{p_0(i)} \sum_j \sqrt{p_0(j)} x(j))$$

induces a map from $C_0(\mathbb{R}^+, \mathbb{R}^I)$ to itself. Let μ^0 be the image measure of ν under this map. It is easily seen that under this law, the canonical process W on $C_0(\mathbb{R}^+, \mathbb{R}^I)$ satisfies

$$\begin{aligned} \mathbb{E}^{\mu^0} \left(e^{\sum_{l=1}^k \sum_i \lambda_l(i)(W_{t_l}(i) - W_{t_{l-1}}(i))} \right) &= \\ e^{\frac{1}{2} \sum_{l=1}^k (t_l - t_{l-1}) (\sum_i p_0(i) \lambda_l(i)^2 - (\sum_i p_0(i) \lambda_l(i))^2)} & \quad (27) \end{aligned}$$

for $k \geq 1$, arbitrary non-decreasing sequences $0 = t_0 \leq t_1 \leq \dots \leq t_k$ of length k and arbitrary (complex) $\lambda_l(i)$'s.

As a consequence, Itô's formula holds with

$$dW_t(i) dW_t(j) = dt (\delta_{i,j} p_0(i) - p_0(i) p_0(j)).$$

Our aim is to use Girsanov's theorem to deform the measure and go from the right-hand side in (27) to the right-hand in (26). If the process U_t with values in \mathbb{R}^I is adapted and satisfies some further technical integrability conditions,

$$M_t^U := e^{\int_0^t \sum_i U_s(i) dW_s(i) - \frac{1}{2} \int_0^t (\sum_i p_0(i) U_s(i)^2 - (\sum_i p_0(i) U_s(i))^2) ds}$$

is a martingale. Moreover, by Itô's formula, $dM_t^U = M_t^U \sum_i U_s(i) dW_s(i)$. Then, by Girsanov's theorem, for any $T > 0$, under the measure $d\mu_T^U := M_T^U d\mu^0$ on $C_0([0, T], \mathbb{R}^I)$, the process $W_t(i) - p_0(i) \int_0^t (U_s(i) - \sum_j p_0(j) U_s(j)) ds$, $t \in [0, T]$, has the same law as $W_t(i)$, $t \in [0, T]$, under $d\mu^0$.

Note that for $t \leq T$, M_t^U is a Radon-Nikodym derivative.

$$M_t^U := \left[\frac{d\mu_T^U}{d\mu^0} \right]_t = \mathbb{E}^{\mu^0} \left(\frac{d\mu_T^U}{d\mu^0} \middle| \mathcal{G}_t \right),$$

In general this construction cannot work for infinite T , because μ_T^U and μ^0 become singular. However, T plays only a dummy role : for $T' \leq T$, $d\mu_T^U$ and $d\mu_{T'}^U$ coincide on $\mathcal{G}_{T'}$. So it is only a slight abuse, which lightens notations a bit, to write $d\mu^U$ for $d\mu_T^U$ and

$$M_t^U = \left[\frac{d\mu^U}{d\mu^0} \right]_t = \mathbb{E}^{\mu^0} \left(\frac{d\mu^U}{d\mu^0} \middle| \mathcal{G}_t \right).$$

For the special choice $U_t(i) := \Gamma(i|\alpha)$, using $\sum_i p_0(i) \Gamma(i|\alpha) = 0$, we compute

$$M_t(\alpha) := e^{\sum_i \Gamma(i|\alpha) W_t(i) - \frac{t}{2} \sum_i p_0(i) \Gamma(i|\alpha)^2}$$

which is certainly a martingale, and we obtain for every T a measure $d\mu_\alpha$ on $C_0([0, T], \mathbb{R}^I)$ such that

$$\mathbb{E}^{\mu_\alpha} \left(e^{\sum_{l=1}^k \sum_i \lambda_l(i) (W_{t_l}(i) - W_{t_{l-1}}(i))} \right) = e^{\sum_{l=1}^k (t_l - t_{l-1}) \sum_i \lambda_l(i) p_0(i) \Gamma(i|\alpha)} e^{\frac{1}{2} \sum_{l=1}^k (t_l - t_{l-1}) (\sum_i p_0(i) \lambda_l(i)^2 - (\sum_i p_0(i) \lambda_l(i))^2)} \quad (28)$$

for $k \geq 1$, arbitrary non-decreasing sequences $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq T$ of length k and arbitrary (complex) $\lambda_l(i)$'s.

Finally, setting $M_t := \sum_\alpha q_0(\alpha) M_t(\alpha)$ (trivially a martingale again) and $d\mu := M_t d\mu^0$ we obtain for every T a measure $d\mu$ on $C_0([0, T], \mathbb{R}^I)$ such that

$$\mathbb{E}^\mu \left(e^{\sum_{l=1}^k \sum_i \lambda_l(i) (W_{t_l}(i) - W_{t_{l-1}}(i))} \right) = \left(\sum_\alpha q_0(\alpha) e^{\sum_{l=1}^k (t_l - t_{l-1}) \sum_i \lambda_l(i) p_0(i) \Gamma(i|\alpha)} \right) \times e^{\frac{1}{2} \sum_{l=1}^k (t_l - t_{l-1}) (\sum_i p_0(i) \lambda_l(i)^2 - (\sum_i p_0(i) \lambda_l(i))^2)} \quad (29)$$

for $k \geq 1$, arbitrary non-decreasing sequences $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq T$ of length k and arbitrary (complex) $\lambda_l(i)$'s. So we have found the continuous time limit of the counting process as the canonical process on $C_0([0, T], \mathbb{R}^I)$ with the measure μ .

It remains to deal with the continuous time limit of the Q_n 's.

We note that by the chain rule $\frac{M_t(\alpha)}{M_t} = \left[\frac{d\mu(\alpha)}{d\mu} \right]_t = \mathbb{E}^\mu \left(\frac{d\mu(\alpha)}{d\mu} \middle| \mathcal{G}_t \right)$ is the Radon-Nikodym derivative of $\mu(\alpha)$ with respect to μ . So with some memory of what happened in the discrete case, it is natural to define $Q_t(\alpha) := q_0(\alpha) \frac{M_t(\alpha)}{M_t}$.

Using the explicit formula for the Q_n 's in terms of the counting processes, we define $Q_t^{(\delta)}$ by an interpolation procedure:

$$Q_t^{(\delta)}(\alpha) := q_0(\alpha) \frac{\prod_i p(i|\alpha) W_t^{(\delta)}(i) / \sqrt{\delta} + p_0(i)t/\delta}{\sum_\beta q_0(\beta) \prod_i p(i|\beta) W_t^{(\delta)}(i) / \sqrt{\delta} + p_0(i)t/\delta},$$

so that if $t/\delta = n$, an integer, $Q_t^{(\delta)}(\alpha) = Q_n(\alpha)$. Note that in this formula the $p(i|\beta)$'s depend implicitly on δ via (14).

One can prove that the joint finite dimensional distributions of the processes $(W_t^{(\delta)}, Q_t^{(\delta)})$ under $\mathbb{P}^{(\delta)}$ have limits when $\delta \rightarrow 0^+$ and that these limits are nothing but the joint finite dimensional distributions of the processes (W_t, Q_t) under μ . In this precise sense, the Q_n 's continuous time limit is deciphered.

This result is really no big surprise, but to prove it we have to rely on an *ad hoc* trick and an explicit elementary but tedious computation. The details are neither illuminating nor elegant so we omit them.

When δ is small enough, all non-empty sets in \mathcal{F}_n have strictly positive measure, so that if $q_0(\alpha) > 0$ the same is true for $Q_n(\alpha)$ for all n 's. As furthermore $\sum_{\alpha} Q_n(\alpha) = 1$ for all n , all the information on the Q_n 's (joint) laws is embodied in the joint laws of ratios of Q_n 's.

As these ratios have a simple product structure in terms of the counting processes, the explicit computation of

$$\mathbb{E} \left(e^{\sum_{l=1}^k \sum_i \lambda_l(i)(N_{n_l}(i) - N_{n_{l-1}}(i))} \prod_{l=1}^k \prod_{\alpha, \beta} \left(\frac{Q_{n_l}(\alpha)}{Q_{n_l}(\beta)} \right)^{\eta_l(\alpha, \beta)} \right)$$

for $k \geq 1$, arbitrary non-decreasing sequences of integers $0 = n_0 \leq n_1 \leq \dots \leq n_k$ of length k , and arbitrary (complex) $\lambda_l(i)$'s and $\eta_l(\alpha, \beta)$'s is in some sense a special case of (18).

The same remark applies to computations involving ratios of Q_t 's.

This allows to compute explicitly that

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \mathbb{E}^{(\delta)} \left(e^{\sum_{l=1}^k \sum_i \lambda_l(i)(W_{t_l}^{(\delta)}(i) - W_{t_{l-1}}^{(\delta)}(i))} \prod_{l=1}^k \prod_{\alpha, \beta} \left(\frac{Q_{t_l}^{(\delta)}(\alpha)}{Q_{t_l}^{(\delta)}(\beta)} \right)^{\eta_l(\alpha, \beta)} \right) = \\ \mathbb{E}^{\mu} \left(e^{\sum_{l=1}^k \sum_i \lambda_l(i)(W_{t_l}(i) - W_{t_{l-1}}(i))} \prod_{l=1}^k \prod_{\alpha, \beta} \left(\frac{Q_{t_l}(\alpha)}{Q_{t_l}(\beta)} \right)^{\eta_l(\alpha, \beta)} \right) \quad (30) \end{aligned}$$

for $k \geq 1$, arbitrary non-decreasing sequences $0 = t_0 \leq t_1 \leq \dots \leq t_k$ of length k , and arbitrary (complex) $\lambda_l(i)$'s and $\eta_l(\alpha, \beta)$'s.

As such a mixture of Laplace and Mellin transforms characterizes the distributions completely, this concludes the existence of a natural continuous time limit.

C Details on the continuous time limit with different partial measurement methods

We use the linear interpolation of appendix B on $W_t^{(\delta)}(o, i) := \sqrt{\delta}(N_{t/\delta}(o, i) - c(o)p_0^o(i)t/\delta)$ if t/δ is an integer. Explicitly for $t \in [\delta n, \delta(n+1)]$,

$$W_t^{(\delta)}(o, i) = \sqrt{\delta}((n+1 - t/\delta)N_n(o, i) + (t/\delta - n)N_{n+1}(o, i) - c(o)p_0^o(i)t/\delta).$$

We remind that $E = \bigcup_{o \in \mathcal{O}} \{o\} \otimes \text{spec}(o)$ is the set of all possible measurement methods and outcomes. We expect the limit time continuous process to live on the vector space of continuous function from \mathbb{R}^+ to \mathbb{R}^E .

Compare to previous sections, the main changes are in the correlation functions calculations. Thanks to the measurement method distribution time

and realization independency, we find:

$$\mathbb{E} \left(e^{\sum_{l=1}^k \sum_{(o,i) \in E} \lambda_l(o,i) (N_{n_l}(o,i) - N_{n_{l-1}}(o,i))} \right) = \sum_{\alpha} q_0(\alpha) \prod_{l=1}^k \left(\sum_{(o,i) \in E} e^{\lambda_l(o,i)} c(o) p^o(i|\alpha) \right)^{n_l - n_{l-1}} \quad (31)$$

for $k \geq 1$, arbitrary non-decreasing sequences of integers $0 = n_0 \leq n_1 \leq \dots \leq n_k$ of length k , and arbitrary (complex) $\lambda_l(o,i)$'s.

As a consequence, in the limit $\delta \rightarrow 0^+$,

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \mathbb{E}^{(\delta)} \left(e^{\sum_{l=1}^k \sum_{(o,i) \in E} \lambda_l(o,i) (W_{t_l}^{(\delta)}(o,i) - W_{t_{l-1}}^{(\delta)}(o,i))} \right) = \\ \left(\sum_{\alpha} q_0(\alpha) e^{\sum_{l=1}^k (t_l - t_{l-1}) \sum_{(o,i) \in E} \lambda_l(o,i) c(o) p_0^o(i) \Gamma^{(o)}(i|\alpha)} \right) \times \\ e^{\frac{1}{2} \sum_{l=1}^k (t_l - t_{l-1}) (\sum_{(o,i) \in E} c(o) p_0^o(i) \lambda_l(o,i)^2 - (\sum_{(o,i) \in E} c(o) p_0^o(i) \lambda_l(o,i))^2)} \quad (32) \end{aligned}$$

for $k \geq 1$, arbitrary non-decreasing sequences $0 = t_0 \leq t_1 \leq \dots \leq t_k$ of length k and arbitrary (complex) $\lambda_l(o,i)$'s and with $\Gamma^{(o)}(i|\alpha) = 2\text{Im} \left(\frac{\langle i | H_{\alpha}^{(o)} | \Psi^{(o)} \rangle}{\langle i | \Psi^{(o)} \rangle} \right)$.

Then each $W_t^{(\delta)}(o,i)$ under $\mu^{(\delta)}$ converges toward a process $W_t(o,i)$ under μ .

The demonstration is then as in previous section except for notational differences which keep track of the o -dependency of $W_t(o,i)$. As in the previous section, the measure μ^0 is defined as the push-forward measure of ν on \mathbb{R}^E under the linear map

$$y(o,i) := \sqrt{c(o) p_0^o(i)} (x(o,i) - \sqrt{c(o) p_0^o(i)} \sum_{(o',j) \in E} \sqrt{c(o') p_0^{o'}(j)} x(o',j))$$

The martingale M_t is defined by

$$M_t = \sum_{\alpha} q_0(\alpha) e^{\sum_{(o,i) \in E} \Gamma^{(o)}(i|\alpha) W_t(o,i) - \frac{t}{2} \sum_{(o,i) \in E} c(o) p_0^o(i) \Gamma^{(o)}(i|\alpha)^2}$$

The measure μ is defined via Girsanov's transformation: $\mathbb{E}^{\mu}(\cdot) = \mathbb{E}^{\mu^0}(M_t \cdot)$.

D Derivation of the density matrix evolution

Let us derive the density matrix continuous time limit. Recall that at time n its elements are functions of the counting processes

$$A_n(\alpha, \beta) = \frac{A_0(\alpha, \beta) \prod_i (M(i|\alpha) M(i|\beta)^*)^{N_n(i)}}{\sum_{\gamma} q_0(\gamma) \prod_i p(i|\gamma)^{N_n(i)}}$$

from this expression we define time continuous processes

$$A_t^{(\delta)}(\alpha, \beta) = \frac{A_0(\alpha, \beta) \prod_i (M(i|\alpha)M(i|\beta)^*)^{W_t^{(\delta)}(i)/\sqrt{\delta}+p_0(i)\frac{t}{\delta}}}{\sum_{\gamma} q_0(\gamma) \prod_i p(i|\gamma)^{W_t^{(\delta)}(i)/\sqrt{\delta}+p_0(i)\frac{t}{\delta}}}$$

equal to $A_n(\alpha, \beta)$ if $t/\delta = n$ is an integer. The $M(i|\alpha)$'s depend explicitly on δ via

$$M(i|\alpha) = \langle i|e^{-i\delta(E_\alpha \mathbb{I} + H_p + \frac{1}{\sqrt{\delta}}H_\alpha)}|\Psi\rangle$$

We rewrite the products over the partial measurement results as exponentials of sums

$$A_t^{(\delta)}(\alpha, \beta) = \frac{A_0(\alpha, \beta) e^{\sum_i \ln(M(i|\alpha)M(i|\beta)^*/p_0(i))(W_t^{(\delta)}(i)/\sqrt{\delta}+p_0(i)\frac{t}{\delta})}}{\sum_{\gamma} q_0(\gamma) e^{\sum_i \ln(p(i|\gamma)/p_0(i))(W_t^{(\delta)}(i)/\sqrt{\delta}+p_0(i)\frac{t}{\delta})}}$$

A detailed analysis of the limit $\delta \rightarrow 0^+$ would require to perform the same study as in section B. However, at this stage we are confident enough to state that we can safely shortcut a few steps and use directly that $W_t^{(\delta)}(i)$ converge. Using $\langle \Psi|H_I|\Psi\rangle = 0$, and the identity $\sum_i p_0(i) \frac{\langle i|H_\alpha^2|\Psi\rangle}{\langle \Psi|i\rangle} = \sum_i p_0(i) |c(i|\alpha)|^2$, we obtain

$$\begin{aligned} \lim_{\delta \rightarrow 0} \sum_i \ln(M(i|\alpha)M(i|\beta)^*/p_0(i))(W_t^{(\delta)}(i)/\sqrt{\delta}+p_0(i)\frac{t}{\delta}) \\ = l(\alpha, \beta)t - i \sum_i (c(i|\alpha) - c(i|\beta)^*)W_t(i) \end{aligned}$$

where the limit as to be understood as the limit of any finite dimensional correlation functions. Therefore

$$\lim_{\delta \rightarrow 0} A_t^{(\delta)}(\alpha, \beta) = A_t(\alpha, \beta)$$

with $A_t(\alpha, \beta)$ defined in (22). It is then a simple matter, using Itô rules for $W_t(i)$, to derive the Belavkin equation (23) for the density matrix $\rho_t = \sum_{\alpha, \beta} A_t(\alpha, \beta) |\alpha\rangle \langle \beta|$.

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